HIGHER-ORDER ALEXANDER INVARIANTS OF PLANE ALGEBRAIC CURVES

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Abstract. We define new higher-order Alexander modules $\mathcal{A}_n(C)$ and higher-order degrees $\delta_n(C)$ which are invariants of the algebraic planar curve C. These come from analyzing the module structure of the homology of certain solvable covers of the complement of the curve C. These invariants are in the spirit of those developed by T. Cochran in [2] and S. Harvey in [8] and [9], which were used to study knots, 3-manifolds, and finitely presented groups, respectively. We show that for curves in general position at infinity, the higher-order degrees are finite. This provides new obstructions on the type of groups that can arise as fundamental groups of complements to affine curves in general position at infinity.

1. Introduction

The study of singular plane curves is a subject going back to the work of Zariski, who observed that the position of singularities has an influence on the topology of the curve, and that this phenomena can be detected by the fundamental group of the complement. However, the fundamental group of a plane curve complement is in general highly non-commutative, and thus difficult to handle. It is therefore natural to look for invariants of the fundamental group that capture information about the topology of the curve, such as Alexander-type invariants associated to various covering spaces of the curve complement. By analogy with the classical theory of knots and links in a 3-sphere, Libgober developed invariants of the total linking number infinite cyclic cover in [10, 13, 14] and those of the universal abelian cover in [17, 16]. In this paper, we consider certain solvable covers of the curve complement, and their associated Alexander invariants.

Using techniques developed by T. Cochran, K. Orr, and P. Teichner in [3], S. Harvey (in [8]) and T. Cochran (in [2]) defined higher-order Alexander modules and higher-order degrees associated to 3-manifolds and knots, respectively. They used these invariants to show that certain groups cannot be realized as the fundamental group of the complement of a knot, or as the fundamental group of a 3-manifold. In the present paper, we use the same type of invariants to study the complement of complex plane algebraic curves. Our main result shows that under certain restrictions on the curve, these invariants are uniformly bounded. This provides a new obstruction on the groups being realizable as the fundamental group of the complement to a plane curve.

1.1. Survey of results.

Let C be a reduced curve in $\mathbb{C}^2$, and consider $\mathcal{U}$, the complement of C in $\mathbb{C}^2$, with $G = \pi_1(\mathcal{U})$. The multivariable Alexander invariant, studied in [16, 17] (but see also [6]), is defined by considering the universal abelian covering space of $\mathcal{U}$ corresponding

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to the map $G \to G/[G,G] \cong \mathbb{Z}^s$, where s is the number of irreducible components of the curve. We continue this construction by taking iterated universal torsion-free abelian covers of $\mathcal{U}$ corresponding to the maps $G \to G/G_r^{(n+1)} \equiv \Gamma_n$, where $G_r^{(n)}$ is the n^{th}-term in the rational derived series of G (defined in §2 below). We define the higher-order Alexander modules of the plane curve complement to be $\mathcal{A}_n^{\mathbb{Z}}(C) = H_1(\mathcal{U}; \mathbb{Z}\Gamma_n)$, and note that $\mathcal{A}_0^{\mathbb{Z}}(C)$ is just the (integral) universal abelian Alexander module of C. The following is a corollary to Theorem 4.1, and provides an analogue to similar results from the infinite cyclic and universal abelian cases:

Corollary 4.2. *If C is a reduced curve in $\mathbb{C}^2$, that is in general position at infinity, $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module.*

Furthermore, we consider some skew Laurent polynomial rings $\mathbb{K}_n[t^{\pm 1}]$, which are obtained from $\mathbb{Z}\Gamma_n$ by inverting the non-zero elements of a particular subring. The advantage of using $\mathbb{K}_n[t^{\pm 1}]$ coefficients instead of $\mathbb{Z}\Gamma_n$ coefficients is that the former is a principal ideal domain. We define the higher-order degree of C to be $\delta_n(C) = \mathrm{rk}_{\mathbb{K}_n} H_1(\mathcal{U}; \mathbb{K}_n[t^{\pm 1}])$.

Even though, in principle, the higher-order degrees may be computed by means of Fox free calculus (cf. [8], §6), the calculations are tedious as they depend on a presentation of the fundamental group of the curve complement. However, in the case that the curve is in general position at infinity, we find a uniform upper bound on the higher-order degrees. In particular, we prove the following result:

Theorem 4.1. *Suppose C is a degree d curve in $\mathbb{C}^2$, such that its projective completion $\bar{C}$ is transverse to the line at infinity. If C has singularities c_k, $1 \leq k \leq l$, then*

$$\delta_n(C) \leq \Sigma_{k=1}^{l} \left(\mu(C, c_k) + 2n_k\right) + 2g + d - l, \tag{1.1}$$

where $\mu(C, c_k)$ is the Milnor number associated to the singularity germ at c_k, n_k is the number of branches through the singularity c_k, and g is the genus of the normalized curve.

As a direct corollary of the proof of Theorem 4.1, we also find a bound on the higher-order degrees of the curve in terms of "local" degrees, $\bar{\delta}_n^k$, for each singularity c_k of C. The latter were defined and studied by Harvey [8].

Theorem 4.5. *If C satisfies the assumptions of the previous theorem, then*

$$\delta_n(C) \leq \Sigma_{k=1}^{l} (\bar{\delta}_n^k + 2n_k) + 2g + d,$$

where $\bar{\delta}_n^k = \bar{\delta}_n(X_k)$ is Harvey's invariant of the link complement X_k associated to the singularity c_k.

We view Theorem 4.5 as an analogue of the divisibility properties for the infinite cyclic Alexander polynomial of the complement as shown in [13].

For irreducible curves, regardless of the position of the line at infinity, the higher-order degrees are finite and thus the higher-order Alexander modules are torsion. However, if the line at infinity is not transverse to the irreducible curve C, then the upper bounds mentioned above will also include the contribution of the singular points at infinity (similar to [12], Theorem 4.3).

To complete the analogy with the case of Alexander polynomials of the infinite cyclic cover of the complement, we also provide an upper bound on $\delta_n(C)$ by the corresponding higher-order Alexander invariant of the link at infinity (see Theorem 4.7). For a curve of degree d, in general position at infinity, this is an uniform bound equal to $d(d-2)$.

We conclude the paper with sample computations of higher-order degrees of plane curves, also indicating some connections with the infinite cyclic and universal abelian invariants of the curve. In passing we note that the higher-order degrees δ_n, at any level n, are sensitive to the position of singular points (see Example 5.6). This fact alone gives an incentive to look for examples of Zariski pairs that are distinguished by some δ_k, but not by any other Alexander-type invariants. This will make the subject of future work by these authors.

1.2. **Connections with other work.** We end the introduction with presenting further motivation for this work and connections with previous studies on invariants of the fundamental group of a plane curve complement.

Although in geometric problems the fundamental group of complements to projective curves plays a central role, by switching to the affine setting (i.e. by removing also a generic line) no essential information is lost. Indeed, if $\bar{C}$ is the projective completion of C and H is the line at infinity, the two groups are related by the central extension

$$0 \to \mathbb{Z} \to \pi_1(\mathbb{CP}^2 - (\bar{C} \cup H)) \to \pi_1(\mathbb{CP}^2 - \bar{C}) \to 0.$$

Moreover, as Oka proved (e.g., see [23], Lemma 2), the commutators of the two fundamental groups (in the affine and resp. projective setting) coincide, therefore either of the two groups can be used for computing the rational derived series of the fundamental group of the affine complement.

Our finiteness result on the higher-order degrees provides also new obstructions on the type of groups that can arise as fundamental groups of complements to affine hypersurfaces in general position at infinity. Indeed, by a Zariski-Lefschetz theorem, possible fundamental groups of complements to hypersurfaces in $\mathbb{C}^n$ are precisely the fundamental groups of affine plane curve complements. Note that for a general group, one does not expect the higher-order degrees δ_n to be finite. For instance, for a free group with at least 2 generators the free ranks r_n are positive (cf. [8], Example 8.2) therefore δ_n is infinite.

Similar obstructions on fundamental groups of complements to affine plane curves were previously obtained by Libgober and others (for a nice discussion on this topic in relation with a question of Serre, the reader is advised to consult [15]). For example, from the study of the total linking number infinite cyclic cover of the complement [10, 13], it follows that the Alexander polynomial of the (affine) curve is cyclotomic. More precisely, for a curve in general position at infinity this polynomial divides the product of the local Alexander polynomials at the singular points, and its zeros are roots of unity of order $d = \deg(\bar{C})$. This result already obstructs many knot groups from being realizable as fundamental groups of complements to affine plane curves. More obstructions were derived by Libgober [16, 17] and Arapura [1] from the study of the universal abelian cover of the affine complement. We only mention here the powerful result of Arapura which states that the support (hence all characteristic varieties) of the fundamental group of a plane curve complement is a union of subtori of the character torus, possibly translated by unitary characters.

Our obstructions come from analyzing the solvable coverings associated to the rational derived series of the fundamental group of the affine complement. It would be interesting at this point to understand how the higher-order degrees are related to (or influenced by) the invariants of the infinite cyclic or universal abelian covers of the complement. Proposition 5.1 already provides such a relation. In connection with the universal abelian cover, Libgober proved that if the codimension (in the character torus) of support of the universal abelian

Alexander module is greater than 1, then $\delta_0(C) = 0$. Of course, this assumption can only be satisfied if the curve has at least 2 irreducible components, and it remains to understand for what type of curves such a condition holds.

2. Rational derived series of a group; PTFA groups

In this section, we review the definitions and basic constructions that we will need from [8] and [2]. More details can be found in these sources.

We begin by recalling the definition of the rational derived series $\{G_r^{(i)}\}$ associated to any group G.

Definition 2.1. *Let $G_r^{(0)} = G$. For $n \geq 1$, define the n^{th} term of the rational derived series of G by:*

$$G_r^{(n)} = \{g \in G_r^{(n-1)} | g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z} - \{0\}\}.$$

We denote by Γ_n the quotient $G/G_r^{(n+1)}$. Since $G_r^{(n)}$ is a normal subgroup of $G_r^{(i)}$ for $0 \leq i \leq n$ ([8], Lemma 3.2), it follows that Γ_n is a group.

The use of rational derived series, as opposed to the usual derived series $\{G^{(n)}\}$, is necessary to avoid zero divisors in the group ring $\mathbb{Z}\Gamma_n$. However, if G is a knot group or a free group, the rational derived series and the derived series coincide ([8], p. 902). If G is a finite group then $G_r^{(n)} = G$, hence $\Gamma_n = \{1\}$ for all $n \geq 0$.

The rational derived series is defined in such a way that the successive quotients $G_r^{(n)}/G_r^{(n+1)}$ are $\mathbb{Z}$-torsion-free and abelian. In fact ([8], Lemma 3.5):

$$G_r^{(n)}/G_r^{(n+1)} \cong \left(G_r^{(n)}/[G_r^{(n)}, G_r^{(n)}]\right)/\{\mathbb{Z} - \text{torsion}\}. \tag{2.1}$$

If $G = \pi_1(X)$ this says that $G_r^{(n)}/G_r^{(n+1)} \cong H_1(X_{\Gamma_{n-1}})/\{\mathbb{Z} - \text{torsion}\}$, where $X_{\Gamma_{n-1}}$ is the regular Γ_{n-1}-cover of X. In particular, $G/G_r^{(1)} = G_r^{(0)}/G_r^{(1)} \cong H_1(X)/\{\mathbb{Z} - \text{torsion}\} \cong \mathbb{Z}^{\beta_1(X)}$.

Remark 2.2. If G is the fundamental group of a link complement in S^3 or that of a plane curve complement, then $G_r^{(1)} = G^{(1)}$ (since there is no torsion in the first homology of the complement).

Definition 2.3. *A group Γ is poly-torsion-free-abelian (PTFA) if it admits a normal series of subgroups $\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = \Gamma$ such that each of the successive quotients G_{i+1}/G_i is torsion-free abelian.*

Remark 2.4. We collect here the following facts:

(1) Any subgroup of a PTFA group is a PTFA group.

(2) If Γ is PTFA, then $\mathbb{Z}\Gamma$ is a right (and left) Ore domain (i.e., has no zero divisors and $\mathbb{Z}\Gamma - \{0\}$ is a right divisor set). Thus it embeds in its classical right ring of quotients $\mathcal{K}$, a skew field ([2], Proposition 3.2).

(3) If R is an Ore domain and S is a right divisor set, then RS^{-1} is flat as a left R-module. In particular, $\mathcal{K}$ is a flat $\mathbb{Z}\Gamma$-module ([24], Proposition II.3.5).

(4) Every module over $\mathcal{K}$ is a free module ([24], Proposition I.2.3). Such modules have a well-defined rank $\mathrm{rk}_{\mathcal{K}}$ which is additive on short exact sequences.

If A is a module over an Ore domain R, then the rank of A is defined as $rk(A) = \mathrm{rk}_{\mathcal{K}}(A \otimes_R \mathcal{K})$, where $\mathcal{K}$ is the quotient field of R. In particular, A is a torsion R-module if and only if $A \otimes_R \mathcal{K} = 0$.

Proposition 2.5. *([8], Corollary 3.6) For any group G, $\Gamma_n = G/G_r^{(n+1)}$ is PTFA. Thus $\mathbb{Z}\Gamma_n$ embeds in its classical right ring of quotients, $\mathcal{K}_n$.*

Suppose X is a topological space that has the homotopy type of a connected CW-complex. Let Γ be any group and $\phi : \pi_1(X) \to \Gamma$ be a homomorphism. We denote by X_Γ the regular Γ-cover of X associated to ϕ. If ϕ is surjective, this is the covering space associated to $\ker \phi$. (For details about the case where ϕ is not surjective, we refer the reader to §3 of [2].) Let $C(X_\Gamma; \mathbb{Z})$ be the $\mathbb{Z}\Gamma$-free cellular chain complex for X_Γ obtained by lifting the cell structure of X. If $\mathcal{M}$ is a $\mathbb{Z}\Gamma$-bimodule, then define:

$$H_*(X; \mathcal{M}) = H_*(C(X_\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} \mathcal{M})$$

as a right $\mathbb{Z}\Gamma$-module.

Proposition 2.6. *([3], Proposition 2.9) Let X be a connected CW-complex and Γ a PTFA group. If $\phi : \pi_1(X) \to \Gamma$ is a non-trivial coefficient system, then $H_0(X; \mathbb{Z}\Gamma)$ is a torsion $\mathbb{Z}\Gamma$-module.*

Proposition 2.7. *([2], Proposition 3.10) Let X be a connected CW-complex and Γ be a PTFA group. Suppose $\pi_1(X)$ is finitely generated and $\phi : \pi_1(X) \to \Gamma$ is a non-trivial coefficient system. Then $rk\,(H_1(X; \mathbb{Z}\Gamma)) \leq \beta_1(X) - 1$, where $\beta_1(X)$ is the first Betti number of X. In particular, if $\beta_1(X) = 1$ then $H_1(X; \mathbb{Z}\Gamma)$ is a torsion $\mathbb{Z}\Gamma$-module.*

3. Definitions of new invariants

Let C be a reduced curve in $\mathbb{C}^2$, defined by the equation: $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f, and let $C_i = \{f_i = 0\}$ denote the irreducible components of C. Embed $\mathbb{C}^2$ in $\mathbb{CP}^2$ by adding the plane at infinity, H, and let $\bar{C}$ be the projective curve in $\mathbb{CP}^2$ defined by the homogenization f^h of f. We let $\bar{C}_i = \{f_i^h = 0\}$, $i = 1, \cdots, s$, be the corresponding irreducible components of $\bar{C}$. Let $\mathcal{U}$ be the complement $\mathbb{CP}^2 - (\bar{C} \cup H)$. (Alternatively, $\mathcal{U}$ may be regarded as the complement of the curve C in the affine space $\mathbb{C}^2$.) Then $H_1(\mathcal{U})$ is free abelian generated by the meridian loops γ_i about the non-singular part of each irreducible component $\bar{C}_i$, for $i = 1, \cdots, s$ (cf. [4], (4.1.3), (4.1.4)). If γ_∞ denotes the meridian about the line at infinity, then the equation $\gamma_\infty + \sum_{i=1}^{s} d_i \gamma_i = 0$ with $d_i = deg(f_i^h)$, holds in $H_1(\mathcal{U})$.

3.1. **Higher-order Alexander modules.** We let $G = \pi_1(\mathcal{U})$, $\Gamma_n = G/G_r^{(n+1)}$, and $\mathcal{K}_n$ be the classical right ring of quotients of $\mathbb{Z}\Gamma_n$.

Definition 3.1. *We define the higher-order Alexander modules of the plane curve to be:*

$$\mathcal{A}_n^{\mathbb{Z}}(C) = H_1(\mathcal{U}; \mathbb{Z}\Gamma_n) = H_1(\mathcal{U}_{\Gamma_n}; \mathbb{Z})$$

where $\mathcal{U}_{\Gamma_n}$ is the covering of $\mathcal{U}$ corresponding to the subgroup $G_r^{(n+1)}$. That is, $\mathcal{A}_n^{\mathbb{Z}}(C) = G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}]$ as a right $\mathbb{Z}\Gamma_n$-module.

Definition 3.2. *The n^{th} order rank of (the complement of) C is:*

$$r_n(C) = rk\mathcal{A}_n^{\mathbb{Z}}(C).$$

Remark 3.3. (1) Note that $\mathcal{A}_0^{\mathbb{Z}}(C) = G_r^{(1)}/[G_r^{(1)}, G_r^{(1)}] = G'/G''$, by Remark 2.2. This is just the universal abelian invariant of the complement.

(2) $\mathcal{A}_n^{\mathbb{Z}}(C)/\{\mathbb{Z}-\text{torsion}\} = G_r^{(n+1)}/G_r^{(n+2)}$.

(3) If C is irreducible, then $\beta_1(\mathcal{U}) = 1$. By Proposition 2.7, it follows that $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion module.

In Corollary 4.2, we show that under the assumption of transversality at infinity, the module $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module. Therefore, $r_n(C) = 0$.

Since $\mathcal{U}$ is a 2-dimensional affine variety, it has the homotopy type of a 2-dimensional CW-complex. Thus the modules $H_k(\mathcal{U}; \mathbb{Z}\Gamma_n)$ are trivial for $k > 2$ and $H_2(\mathcal{U}; \mathbb{Z}\Gamma_n)$ is a torsion-free $\mathbb{Z}\Gamma_n$-module. Moreover, we will show that in our setting, the rank of $H_2(\mathcal{U}; \mathbb{Z}\Gamma_n)$ is equal to the Euler characteristic of the complement, $\mathcal{U}$.

Remark 3.4. Assume that the universal abelian Alexander module of the complement is *trivial*, i.e. $\mathcal{A}_0^{\mathbb{Z}}(C) = 0$. (Note that this is the case if G is abelian or finite.) Then all higher-order Alexander modules $\mathcal{A}_n^{\mathbb{Z}}(C) = 0$, for $n \geq 1$, are also trivial. Indeed, by Remark 2.2, $G' = G_r^{(1)}$ and $\mathcal{A}_0^{\mathbb{Z}}(C) = G'/G''$. It follows that $G^{(n)} = G' = G_r^{(1)}$, for all $n \geq 1$. From the definition of the rational derived series, it is now easy to see that $G_r^{(n)} = G'$ for all $n \geq 1$. Therefore $\mathcal{A}_n^{\mathbb{Z}}(C) \cong G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}] \cong G'/G'' = 0$, for all $n \geq 0$.

Example 3.5. (1) If C is a non-singular curve in general position at infinity, then $\pi_1(\mathcal{U}) \cong \mathbb{Z}$ (cf. [12]), hence abelian. By the above remark, it follows that $\mathcal{A}_n^{\mathbb{Z}}(C) = 0$, for all $n \geq 0$.
(2) Suppose $\mathcal{U}$ is the complement in $\mathbb{C}^2$ of a union of two lines. Then $\pi_1(\mathcal{U})$ is $\mathbb{Z}^2$. Hence $\mathcal{A}_n^{\mathbb{Z}}(C) = 0$ for all $n \geq 0$.
(3) If $\bar{C}$ is a reduced curve having only nodes as its singularities (i.e., locally at each singular point, $\bar{C}$ looks like $x^2 - y^2 = 0$), then it is known that $\pi_1(\mathbb{CP}^2 - \bar{C})$ is abelian (e.g., see [23]), thus has trivial commutator subgroup. Under the assumption that the line at infinity is generic, this implies that the commutator subgroup of $\pi_1(\mathcal{U})$ is also trivial ([23], Lemma 2), so $\pi_1(\mathcal{U})$ is abelian. Now from Remark 3.4 it follows that $\mathcal{A}_n^{\mathbb{Z}}(C) = 0$, for all $n \geq 0$.

3.2. **Localized higher-order Alexander modules.** In this section we define some skew Laurent polynomial rings $\mathbb{K}_n[t^{\pm 1}]$, which are obtained from $\mathbb{Z}\Gamma_n$ by inverting the non-zero elements of a particular subring described below. This construction is used in [8] and [2] and is described in algebraic generality in [9]. We refer to those sources for the background definitions.

Recall our notations: $G = \pi_1(\mathcal{U})$, $\Gamma_n = G/G_r^{(n+1)}$ and $\mathcal{K}_n$ is the classical right ring of quotients of $\mathbb{Z}\Gamma_n$. Let $\psi \in H^1(G; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$ be the primitive class representing the linking number homomorphism $G \xrightarrow{\psi} \mathbb{Z}$, $\alpha \mapsto \mathrm{lk}(\alpha, C)$. Since the commutator subgroup of G is in the kernel of ψ, it follows that ψ induces a well-defined epimorphism $\bar{\psi} : \Gamma_n \to \mathbb{Z}$. Let $\bar{\Gamma}_n$ be the kernel of $\bar{\psi}$. Since $\bar{\Gamma}_n$ is a subgroup of Γ_n, by Remark 2.4, $\bar{\Gamma}_n$ is a PTFA group. Thus $\mathbb{Z}\bar{\Gamma}_n$ is an Ore domain and $S_n = \mathbb{Z}\bar{\Gamma}_n - \{0\}$ is a right divisor set of $\mathbb{Z}\bar{\Gamma}_n$. Let $\mathbb{K}_n = (\mathbb{Z}\bar{\Gamma}_n)S_n^{-1}$ be the right ring of quotients of $\mathbb{Z}\bar{\Gamma}_n$, and set $R_n = (\mathbb{Z}\Gamma_n)S_n^{-1}$.

If we *choose* a $t \in \Gamma_n$ such that $\bar{\psi}(t) = 1$, this yields a splitting $\phi : \mathbb{Z} \to \Gamma_n$ of $\bar{\psi}$. As in Proposition 4.5 of [8], the embedding $\mathbb{Z}\bar{\Gamma}_n \hookrightarrow \mathbb{K}_n$ extends to an isomorphism $R_n \xrightarrow{\cong} \mathbb{K}_n[t^{\pm 1}]$. (However this isomorphism depends on the choice of splitting!) It follows that R_n is a non-commutative principal left and right ideal domain, since this is known to be true for any skew Laurent polynomial rings with coefficients in a skew field ([2], Proposition 4.5). Also note that by Remark 2.4, R_n is a flat left $\mathbb{Z}\Gamma_n$-module.

Definition 3.6. *The n^{th}-order localized Alexander module of the curve C is defined to be $\mathcal{A}_n(C) = H_1(\mathcal{U}; R_n)$, viewed as a right R_n-module. If we choose a splitting ϕ to identify R_n with $\mathbb{K}_n[t^{\pm 1}]$, we define $\mathcal{A}_n^{\phi}(C) = H_1(\mathcal{U}; \mathbb{K}_n[t^{\pm 1}])$.*

Definition 3.7. *The n^{th}-order degree of C is defined to be:*

$$\delta_n(C) = rk_{\mathbb{K}_n} \mathcal{A}_n(C)$$

Remark 3.8. For any choice of ϕ, $\mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n(C) = \mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n^{\phi}(C)$. So although the module $\mathcal{A}_n^{\phi}(C)$ depends on the splitting, the rank of the module does not.

The degrees $\delta_n(C)$ are integral invariants of the fundamental group G of the complement. Indeed, we have (cf. [9], §1):

$$\delta_n(C) = rk_{\mathbb{K}_n} \left(G_r^{(n+1)} / [G_r^{(n+1)}, G_r^{(n+1)}] \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n \right) \tag{3.1}$$

Furthermore, for any choice of splitting ϕ, since $\mathbb{K}_n[t^{\pm 1}]$ is a principal ideal domain, there exist some nonzero $p_i(t) \in \mathbb{K}_n[t^{\pm 1}]$, $i = 1, \cdots, m$, such that:

$$\mathcal{A}_n^{\phi}(C) \cong \left(\oplus_{i=1}^m \frac{\mathbb{K}_n[t^{\pm 1}]}{p_i(t)\mathbb{K}_n[t^{\pm 1}]} \right) \oplus \mathbb{K}_n[t^{\pm 1}]^{r_n(C)}$$

Therefore, $\delta_n(C)$ is finite if and only if one of the equivalent statements is true:

(1) $r_n(C) = \mathrm{rk}_{\mathcal{K}_n} H_1(\mathcal{U}; \mathcal{K}_n) = 0$.
(2) $\mathcal{A}_n(C)$ is a torsion R_n-module.
(3) For any ϕ, $\mathcal{A}_n^{\phi}(C)$ is a torsion $\mathbb{K}_n[t^{\pm 1}]$-module.
(4) $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module.

If this is the case, then $\delta_n(C)$ is the sum of the degrees of the polynomials $p_i(t)$.

Remark 3.9. It is interesting to note that if C is irreducible, then $\delta_0(C)$ is the degree of the Alexander polynomial of C. (The latter was defined and studied by Libgober in a sequence of papers [10, 11, 12, 13, 14].) Indeed, this follows directly from the above definition or from (3.1), since in the irreducible case we have that $\bar{\Gamma}_0 \cong 0$, therefore $\mathbb{K}_0 \cong \mathbb{Q}$.

The invariant $\delta_n(C)$ is difficult to calculate, in general. However, the special case of weighted homogeneous affine curves is well understood:

Proposition 3.10. *Suppose C is defined by a weighted homogeneous polynomial $f(x, y) = 0$ in $\mathbb{C}^2$, and assume that either $n > 0$ or $\beta_1(\mathcal{U}) > 1$. Then we have:*

$$\delta_n(C) = \mu(C, 0) - 1, \tag{3.2}$$

where $\mu(C, 0)$ is the Milnor number associated to the singularity germ at the origin. If $\beta_1(\mathcal{U}) = 1$, then $\delta_0(C) = \mu(C, 0)$.

Proof. The key observation here is the existence of a global Milnor fibration (see for example [4], (3.1.12)):

$$F = \{f = 1\} \hookrightarrow \mathcal{U} = \mathbb{C}^2 - C \xrightarrow{f} \mathbb{C}^*,$$

and the fact that F is homotopy equivalent to the infinite cyclic cover of $\mathcal{U}$ corresponding to the kernel of the total linking number homomorphism ψ. The Γ_n-cover of $\mathcal{U}$ factors through the infinite cyclic cover corresponding to ψ, which is homotopy equivalent to F. It follows that there is an isomorphism of $\mathbb{K}_n$-modules:

$$H_*(\mathcal{U}; R_n) \cong H_*(F; \mathbb{K}_n).$$

In particular,

$$\delta_n(C) = \mathrm{rk}_{\mathbb{K}_n} H_1(F; \mathbb{K}_n).$$

Since F has the homotopy type of a 1-dimensional CW complex, $H_2(F; \mathbb{K}_n) = 0$. Moreover, if either $n > 0$ or $\beta_1(\mathcal{U}) > 1$, the coefficient system $\pi_1(F) \to \bar{\Gamma}_n$ is non-trivial. Hence, by Proposition 2.6, $H_0(F; \mathbb{K}_n) = 0$. It follows, in this case, that

$$\delta_n(C) = -\chi(F) = \mu(C, 0) - 1.$$

On the other hand, if $\beta_1(\mathcal{U}) = 1$, then $\mathrm{rk}_{\mathbb{K}_0} H_0(F; \mathbb{K}_0) = \mathrm{rk}_{\mathbb{Q}} H_0(F; \mathbb{Q}) = 1$. Hence, if $\beta_1(\mathcal{U}) = 1$, then $\delta_0(C) = 1 - \chi(F) = \mu(C, 0)$.

□

Example 3.11. Since $f(x) = x^3 - y^2$ is a weighted homogeneous polynomial, if C is the curve defined by $f = 0$, it follows from Proposition 3.10, that $\delta_0(C) = 2$ and $\delta_n(C) = 1$ for $n > 0$.

Remark 3.12. Due to the existence of Milnor fibrations, we note that formula (3.2) holds for the case of any algebraic link, by replacing $\mathcal{U}$ by the link complement and $\delta_n(C)$ by Harvey's invariant of the algebraic link. For a more general discussion on fibered 3-manifolds, see [8], Proposition 8.4, 8.5.

Remark 3.13. As noted in [8], §6 and §8, the higher-order Alexander invariants $r_n(C)$ and $\delta_n(C)$ can be computed from a presentation of the fundamental group of the curve complement, by means of Fox free calculus.

4. Upper bounds on the higher order degree of a curve complement

In this section, we find upper bounds for $\delta_n(C)$. In Theorem 4.1, we find an upper bound in terms of the Milnor number of each singularity. In Theorem 4.5, we express this bound in terms of the Harvey's invariants, $\bar{\delta}_n$, associated to each of the singular points of C. This result is analogous to the divisibility properties for the infinite cyclic Alexander polynomial of the complement (e.g., see [11, 12, 13, 19]). As a corollary to these theorems, we have that, if C is a curve in general position at infinity, then $\delta_n(C)$ is finite, and therefore $\mathcal{A}_n(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module. We also give an upper bound for $\delta_n(C)$ in terms of the higher-order degrees of the link at infinity.

Theorem 4.1. *Suppose C is a degree d curve in $\mathbb{C}^2$ such that its projective completion $\bar{C}$ is transverse to the line at infinity H. If C has singularities c_k, $1 \leq k \leq l$, then*

$$\delta_n(C) \leq \Sigma_{k=1}^{l} \left(\mu(C, c_k) + 2n_k\right) + 2g + d - l, \tag{4.1}$$

where $\mu(C, c_k)$ is the Milnor number associated to the singularity germ at c_k, n_k is the number of branches through the singularity c_k, and g is the genus of the normalized curve.

Before proving Theorem 4.1, we state an immediate corollary.

Corollary 4.2. *If C is a plane curve in general position at infinity, then $\delta_n(C) < \infty$, i.e., $\mathcal{A}^{\mathbb{Z}}_n(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module.*

Remark 4.3. Note that the upper bound in (4.1) is independent of n.

Remark 4.4. If C is an irreducible curve, then independently of the position of the line at infinity, we have that $\beta_1(\mathcal{U}) = 1$. By Proposition 2.7, it follows that $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion module. However, if the curve C is not in general position at infinity, then the upper bound on $\delta_n(C)$ also includes the contribution of the 'singularities at infinity' (similar to Theorem 4.3 of [12]).

Proof. We first reduce the problem to the study of the boundary, X, of a regular neighborhood of C in $\mathbb{C}^2$. In order to do this, let $N(\bar{C})$ be a regular neighborhood of $\bar{C}$ inside $\mathbb{CP}^2$ and note that, due to the transversality assumption, the complement $N(\bar{C})-(\bar{C}\cup H)$ can be identified with $N(C)-C$, where $N(C)$ is a regular neighborhood of C in $\mathbb{C}^2$. But $N(C)-C$ retracts by deformation on $X = \partial N(C)$. Now by the Lefschetz hyperplane section theorem ([4], page 25), it follows that the inclusion map induces a group epimorphism

$$\pi_1(X) \twoheadrightarrow \pi_1(U)$$

(the argument used here is similar to the one used in the proof of Theorem 4.3 of [12]). It follows that $\pi_1(X_{\Gamma_n}) \twoheadrightarrow \pi_1(U_{\Gamma_n})$. Hence, $H_1(X;\mathbb{Z}\Gamma_n) \twoheadrightarrow H_1(\mathcal{U};\mathbb{Z}\Gamma_n)$. Since R_n is a flat $\mathbb{Z}\Gamma_n$-module, there is an R_n-module epimorphism $H_1(X;R_n) \twoheadrightarrow H_1(\mathcal{U};R_n)$. Therefore, we have:

$$\delta_n(C) = \mathrm{rk}_{\mathbb{K}_n} H_1(\mathcal{U};R_n) \leq \mathrm{rk}_{\mathbb{K}_n} H_1(X;R_n).$$

Hence, it is sufficient to bound above the right-hand side of the above inequality. This will follow by a Mayer-Vietoris sequence argument.

Let F be the (abstract) surface obtained from C by removing disks $D_1 \cup \cdots \cup D_{n_k}$ around each singular point c_k of C. Let $N = F \times S^1$. The boundary of N is a union of disjoint tori $T_1^k \cup \cdots \cup T_{n_k}^k$ for $k = 1, \cdots, l$, where l is the number of singular points of C. For each singular point c_k of C we let (S_k^3, L_k) be the link pair of c_k, and denote by X_k the link exterior, $S_k^3 - L_k$. Then X is obtained from N by gluing the link exteriors X_k along the tori T_i^k for $i = 1, \cdots, n_k$:

$$X = N \cup_{\sqcup_i T_i^k} (\sqcup_{k=1}^l X_k).$$

The gluing map sends each longitude of L_k to the restriction of a section in N, and each meridian to a fiber of N.

We consider the Mayer-Vietoris sequence in homology associated to the above cover of X and with coefficients in R_n:

$$\begin{aligned}\cdots \to \oplus_{k,i} H_1(T_i^k;R_n) \xrightarrow{\Psi} H_1(N;R_n) \oplus \left(\oplus_{k=1}^l H_1(X_k;R_n)\right) \to H_1(X;R_n) \\ \to \oplus_{k,i} H_0(T_i^k;R_n) \to H_0(N;R_n) \oplus \left(\oplus_{k=1}^l H_0(X_k;R_n)\right) \to H_0(X;R_n) \to 0\end{aligned}$$

From Remark 2.4, we have:

$$\begin{aligned}\mathrm{rk}_{\mathbb{K}_n} H_1(X;R_n) = \mathrm{rk}_{\mathbb{K}_n} H_1(N;R_n) + \Sigma_{k=1}^l \mathrm{rk}_{\mathbb{K}_n} H_1(X_k;R_n) - \Sigma_{k,i} \mathrm{rk}_{\mathbb{K}_n} H_1(T_i^k;R_n) \\ + \mathrm{rk}_{\mathbb{K}_n} \ker(\Psi) + \Sigma_{k,i} \mathrm{rk}_{\mathbb{K}_n} H_0(T_i^k;R_n) - \mathrm{rk}_{\mathbb{K}_n} H_0(N;R_n) \\ - \Sigma_{k=1}^l \mathrm{rk}_{\mathbb{K}_n} H_0(X_k;R_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(X;R_n).\end{aligned} \tag{4.2}$$

Recall that, for each singular point c_k of C, the coefficient system R_n on X_k is induced by the following composition of maps:

$$\mathbb{Z}\pi_1(X_k) \to \mathbb{Z}\pi_1(X) \to \mathbb{Z}\pi_1(\mathcal{U}) \to \mathbb{Z}\Gamma_n \to R_n.$$

Since each X_k fibers over S^1 with Milnor fiber F_k, the Γ_n-cover of X_k factors through the infinite cyclic cover of X_k which is homeomorphic to $F_k \times \mathbb{R}$. Therefore we have the following isomorphisms of $\mathbb{K}_n$-modules:

$$H_*(X_k; R_n) \cong H_*(F_k; \mathbb{K}_n).$$

Since F_k has the homotopy type of a wedge of circles, $H_2(F_k; \mathbb{K}_n) = 0$. Therefore,

$$\chi(F_k) = -\mathrm{rk}_{\mathbb{K}_n} H_1(F_k; \mathbb{K}_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(F_k; \mathbb{K}_n) = -\mathrm{rk}_{\mathbb{K}_n} H_1(X_k; R_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(X_k; R_n).$$

Similar, since $N = F \times S^1$, the Γ_n-cover of N factors through the infinite cyclic cover of N which is homeomorphic to $F \times \mathbb{R}$. So if F_n denotes the corresponding Γ_n-cover of F, then F_n is a non-compact surface and we have $H_2(F; \mathbb{K}_n) = 0$. Therefore:

$$\chi(F) = -\mathrm{rk}_{\mathbb{K}_n} H_1(F; \mathbb{K}_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(F; \mathbb{K}_n) = -\mathrm{rk}_{\mathbb{K}_n} H_1(N; R_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(N; R_n).$$

Finally, for each k and i, we have:

$$0 = \chi(S^1) = -\mathrm{rk}_{\mathbb{K}_n} H_1(S^1; \mathbb{K}_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(S^1; \mathbb{K}_n) = -\mathrm{rk}_{\mathbb{K}_n} H_1(T_i^k; R_n) + \mathrm{rk}_{\mathbb{K}_n} H_0(T_i^k; R_n).$$

Now we can rewrite equation (4.2) as follows:

$$\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) = -\Sigma_{k=1}^l \chi(F_k) - \chi(F) + \mathrm{rk}_{\mathbb{K}_n} \ker(\Psi) + \mathrm{rk}_{\mathbb{K}_n} H_0(X; R_n). \tag{4.3}$$

Since $\pi_1(X) \twoheadrightarrow \pi(U) \twoheadrightarrow \Gamma_n$ is an epimorphism, it follows that the Γ_n-cover of X is connected, thus yielding that $\mathrm{rk}_{\mathbb{K}_n} H_0(X; R_n) = 1$.

Since $\Psi : \oplus_{k,i} H_1(T_i^k; R_n) \to H_1(N; R_n)$, it follows that $\mathrm{rk}_{\mathbb{K}_n} \ker(\Psi) \leq \Sigma_{k,i} \mathrm{rk}_{\mathbb{K}_n} H_1(T_i^k; R_n)$. For each k and i, we have that:

$$\mathrm{rk}_{\mathbb{K}_n} H_1(T_i^k; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_0(T_i^k; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_0(S^1; \mathbb{K}_n) \leq 1,$$

since S^1 is connected. Therefore, $\mathrm{rk}_{\mathbb{K}_n} \ker(\Psi)$ is less than or equal to the number of tori, which is $\Sigma_{k=1}^l n_k$ where n_k is the number of branches through the singularity c_k. From equation (4.3) we have the following:

$$\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) \leq \Sigma_{k=1}^l (-\chi(F_k) + n_k) - \chi(F) + 1. \tag{4.4}$$

Furthermore, $-\chi(F_k) = \mu(C, c_k) - 1$ and $-\chi(F) \leq 2g + \sum_k n_k + d - 1$, where g is the genus of the normalized curve and d is the degree of the curve, i.e. the number of 'punctures at infinity'. It follows that:

$$\delta_n(C) \leq \mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) \leq \Sigma_{k=1}^l \left(\mu(C, c_k) + 2n_k\right) + 2g + d - l.$$

□

As a corollary of the proof of Theorem 4.1, we obtain the following relation between the higher-order degrees of C and the "local" degrees at singular points:

Theorem 4.5. *Suppose C is a degree d curve in $\mathbb{C}^2$, such that its projective completion $\bar{C}$ is transverse to the line at infinity, H. If C has singularities c_k, $1 \leq k \leq l$, then*

$$\delta_n(C) \leq \Sigma_{k=1}^l (\bar{\delta}_n^k + 2n_k) + 2g + d,$$

where $\bar{\delta}_n^k = \bar{\delta}_n(X_k)$ is Harvey's invariant of the link complement X_k associated to the singularity c_k, n_k is the number of branches through the singularity c_k, and g is the genus of the normalized curve.

Proof. We have equation (4.4) in the above proof:

$$\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) \leq \Sigma_{k=1}^{l}(-\chi(F_k) + n_k) - \chi(F) + 1.$$

Furthermore, $-\chi(F) \leq 2g + \sum_k n_k + d - 1$. From Proposition 8.4 of [8], we have

$$\bar{\delta}_n^k = \bar{\delta}_n(X_k) = \begin{cases} -\chi(F_k) & \text{if } n \neq 0 \text{ or } \beta_1(X_k) \neq 1, \\ 1 - \chi(F_k) & \text{if } n = 0 \text{ and } \beta_1(X_k) = 1. \end{cases}$$

In particular, $-\chi(F_k) \leq \bar{\delta}_n^k$, which proves the theorem. □

We can also give a topological estimate for the rank of the torsion-free $\mathbb{Z}\Gamma_n$-module $H_2(\mathcal{U}; \mathbb{Z}\Gamma_n)$:

Corollary 4.6. *If C is a plane curve in general position at infinity, the rank of the torsion-free $\mathbb{Z}\Gamma_n$-module $H_2(\mathcal{U}; \mathbb{Z}\Gamma_n)$ is equal to the Euler characteristic $\chi(\mathcal{U})$ of the curve complement.*

Proof. Let $\mathcal{C}$ be the equivariant complex

$$0 \to C_2 \to C_1 \to C_1 \to 0$$

of free $\mathbb{Z}\Gamma_n$-modules, obtained by lifting the cell structure of $\mathcal{U}$ to $\mathcal{U}_{\Gamma_n}$, the Γ_n-covering of $\mathcal{U}$. Then $\chi(\mathcal{C}) = \chi(\mathcal{U})$. On the other hand,

$$\chi(\mathcal{C}) = \sum_{i=0}^{2} (-1)^i \mathrm{rk}_{\mathcal{K}_n} H_i(\mathcal{C} \otimes_{\mathbb{Z}\Gamma_n} \mathcal{K}_n) = \sum_{i=0}^{2} (-1)^i rk H_i(\mathcal{C}).$$

Therefore, by Proposition 2.6 and Corollary 4.2, it follows that $\chi(\mathcal{C}) = rk H_2(\mathcal{U}; \mathbb{Z}\Gamma_n)$ and the claim follows. □

We end this section by relating the higher-order degrees of a curve C to the higher-order degrees of its link at infinity. We prove the following theorem, similar in flavor to results on the infinite cyclic and universal abelian Alexander invariants (see [11, 12, 13, 6, 19]):

Theorem 4.7. *Let C be an affine plane curve, and let S^3_∞ be a sphere in $\mathbb{C}^2$ of a sufficiently large radius (that is, the boundary of a small tubular neighborhood in $\mathbb{CP}^2$ of the hyperplane H at infinity). Denote by $C_\infty = S^3_\infty \cap C$ the link of C at infinity, and let X_∞ be its complement $S^3_\infty - C_\infty$, with $G_\infty := \pi_1(X_\infty)$.*

We define δ_n^∞ to be the $\mathbb{K}_n$-rank of $H_1(X_\infty; R_n)$, where the coefficient system is induced by the map $\mathbb{Z}G_\infty \to \mathbb{Z}G \to \mathbb{Z}\Gamma_n \to R_n$. Then:

$$\delta_n(C) \leq \delta_n^\infty. \tag{4.5}$$

Proof. We note that there is a group epimorphism $G_\infty \twoheadrightarrow G$. Indeed, X_∞ is homotopy equivalent to $N(H) - (\bar{C} \cup H)$, where $N(H)$ is a tubular neighborhood of H in $\mathbb{CP}^2$ whose boundary is S^3_∞. If L is a generic line in $\mathbb{CP}^2$, which can be assumed to be contained in $N(H)$, then by the Lefschetz theorem, it follows that the composition

$$\pi_1(L - L \cap (\bar{C} \cup H)) \to \pi_1(N(H) - (\bar{C} \cup H)) \to \pi_1(\mathbb{CP}^2 - (\bar{C} \cup H))$$

is surjective, thus proving our claim (this is the same argument as the one used in [12], Theorem 4.5).

It follows that there is a $\mathbb{Z}\Gamma_n$-module epimorphism

$$H_1(X_\infty; \mathbb{Z}\Gamma_n) \twoheadrightarrow H_1(\mathcal{U}; \mathbb{Z}\Gamma_n).$$

Since R_n is a flat $\mathbb{Z}\Gamma_n$-module, we also get a R_n-module epimorphism:

$$H_1(X_\infty; R_n) \twoheadrightarrow H_1(\mathcal{U}; R_n).$$

This proves the inequality (4.5).

□

For a curve in general position at infinity, this yields a uniform upper bound on the higher-order degrees of the curve, which is independent of the local type of singularities and the number of singular points of the curve:

Corollary 4.8. *If C is a curve of degree d, in general position at infinity, then:*

$$\delta_n(C) \leq d(d-2) \ , \quad \textit{for all } n. \tag{4.6}$$

Proof. The claim follows by noting that if C is transverse to the line at infinity, then C_∞ is the Hopf link on d components (i.e., the union of d fibers of the Hopf fibration), thus an algebraic link. By the argument used in the proof of Proposition 3.10, it follows that $\delta_n^\infty = \mu_\infty - 1$, where μ_∞ is the Milnor number associated to the link at infinity. On the other hand, μ_∞ is the degree of the Alexander polynomial of the link at infinity, so it is equal to $d(d-2)+1$ (cf. [13]). The inequality (4.6) follows now from Theorem 4.7.

□

5. Examples

In this section, we calculate the higher-order degrees for some of the classical examples of irreducible curves, including general cuspidal curves, Zariski's sextics with 6 cusps, Oka's curves, and branched loci of generic projections.

We begin with the following:

Proposition 5.1. *Let $C \subset \mathbb{C}^2$ be an irreducible affine curve. Let $G = \pi_1(\mathbb{C}^2 - C)$, and denote by $\Delta_C(t)$ the Alexander polynomial of the curve complement. If $\Delta_C(t) = 1$, then $\delta_n(C) = 0$ for all n. Moreover, in this case, $\mathcal{A}_n^{\mathbb{Z}}(C) \cong \mathcal{A}_0^{\mathbb{Z}}(C)$ as $\mathbb{Z}[G/G']$-modules, for all n.*

Proof. As C is an irreducible affine curve, we have $G/G' \cong \mathbb{Z}$. Hence $G' \cong G'_r$. The Alexander polynomial $\Delta_C(t)$ is the order of the infinite cyclic (and universal abelian) Alexander module of the complement, that is $G'/G'' \otimes \mathbb{Q}$, regarded as a $\mathbb{Q}[\mathbb{Z}]$-module under the action of the covering transformations group G/G' (cf. [10, 13, 14]). Since C is irreducible, the infinite cyclic Alexander module is a torsion $\mathbb{Q}[t, t^{-1}]$-module, regardless of the position of the line at infinity (cf. [10]). In this setting, $\Delta_C(t)$ can be normalized so that $\Delta_C(1) = 1$.

The triviality of the Alexander polynomial means that the universal abelian module $G'/G'' \otimes \mathbb{Q}$ is trivial, i.e. G'/G'' is a torsion abelian group. By (2.1), we obtain:

$$G'_r/G''_r \cong (G'_r/[G'_r, G'_r])/\{\mathbb{Z} - \text{torsion}\} \cong (G'/G'')/\{\mathbb{Z} - \text{torsion}\} \cong 0.$$

Hence $G''_r \cong G'_r = G'$. It follows by induction that $G_r^{(n)} = G'$, for all $n > 0$. Therefore, for any n,

$$\mathcal{A}_n^{\mathbb{Z}}(C) = G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}] \cong G'/G'' = \mathcal{A}_0^{\mathbb{Z}}(C).$$

Now recall that the higher-order degrees of C may be defined by (3.1):

$$\delta_n(C) = \dim_{\mathbb{K}_n}\left(G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}] \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n\right),$$

where $\bar{\Gamma}_n$ is the kernel of $\Gamma_n \xrightarrow{\bar{\psi}} \mathbb{Z} = G/G'$. The map $\bar{\psi}$ is induced by the total linking number homomorphism, which in our setting is just the abelianization map $G \to G/G' = \mathbb{Z}$. It follows that for all n we have: $\Gamma_n = G/G_r^{(n+1)} = G/G'$, $\bar{\Gamma}_n = G'/G_r^{(n+1)} \cong 0$, and $\mathbb{K}_n \cong \mathbb{Q}$. Therefore, for all n,

$$\delta_n(C) = \dim_{\mathbb{Q}}(G'/G'' \otimes_{\mathbb{Z}} \mathbb{Q}) = 0.$$

□

Note that if the commutator subgroup G' is either perfect (i.e. $G' = G''$) or a torsion group, then the Alexander polynomial of the irreducible curve C is trivial. In particular, this is the case if G is abelian. The following examples deal with each of these cases.

Example 5.2. Let $\bar{C} \subset \mathbb{CP}^2$ be an irreducible curve of degree d which has a cusps (these are locally defined by the equation $x^2 = y^3$) and b nodes as the only singularities. If $d > 6a+2b$, then by a result of Nori (cf. [21], but see also [14]), it follows that $\pi_1(\mathbb{CP}^2 - \bar{C})$ is abelian. If we choose a generic line H 'at infinity' and set $C = \bar{C} - H$, then as in 3.5 it follows that $\pi_1(\mathbb{C}^2 - C)$ is also abelian. Hence all higher-order degrees of C vanish.

Proposition 5.3. Let $\bar{C} \subset \mathbb{CP}^2$ be a degree d irreducible cuspidal curve, i.e. it admits as singularities only nodes and cusps. Choose a generic line $H \subset \mathbb{CP}^2$, and set $C := \bar{C} - H$ and $G = \pi_1(\mathbb{C}^2 - C)$. If $d \not\equiv 0 \pmod 6$, then all higher-order degrees of C vanish.

Proof. This follows from Proposition 5.1 combined with Libgober's divisibility results for the Alexander polynomial of a curve complement (see for instance [13], Theorem 4.1). Indeed, the Alexander polynomial of a cusp is $t^2 - t + 1$, that of a node is $t - 1$, and we use the fact that the Alexander polynomial of an irreducible curve C can be normalized so that $\Delta_C(1) = 1$. Moreover, by our assumption of transversality at infinity, all zeros of $\Delta_C(t)$ are roots of unity of order d. □

Here is a more concrete example:

Example 5.4. *Zariski's three-cuspidal quartic.*

Let $\bar{C} \subset \mathbb{CP}^2$ be a quartic curve with three cusps as its only singularities. Choose as above a generic line H, and set $C = \bar{C} - H$. Then the fundamental group of the affine complement is given by:

$$G = \pi_1(\mathbb{C}^2 - C) = \langle a, b \mid aba = bab, a^2 = b^2 \rangle.$$

It is easy to see (using for example a Redemeister-Shreier process, see [18]) that $G' \cong \mathbb{Z}/3\mathbb{Z}$. It follows by Proposition 5.1 that $\delta_n(C) = 0$, for all n. Moreover, the integral higher Alexander modules are given by: $\mathcal{A}_n^{\mathbb{Z}}(C) = \mathbb{Z}/3\mathbb{Z}$, for all n.

Remark 5.5. If $\bar{C} \subset \mathbb{CP}^2$ is an irreducible quartic curve, but not a three-cuspidal quartic, then the fundamental group $\pi_1(\mathbb{CP}^2 - \bar{C})$ is abelian (cf. [4], Proposition 4.3). If H is a generic line, and $C = \bar{C} - H$, then by [23], Lemma 2, it follows that $\pi_1(\mathbb{C}^2 - C)$ is also abelian. Thus all higher-order degrees of such a curve vanish. Based on this observation and the previous example, it follows that the higher-order degrees of any irreducible quartic curve are all zero.

In what follows, we give examples of curves having (some) non-trivial higher-order degrees. The key observation in these examples is the fact that the higher-order degrees of an affine curve are invariants of the fundamental group of the complement, see (3.1).

Example 5.6. *Sextics with six cusps.*

(a) Let $\bar{C} \subset \mathbb{CP}^2$ be a curve of degree 6 with 6 cusps on a conic. Fix a generic line, H, and set $C = \bar{C} - H$. Then $\pi_1(\mathbb{C}^2 - C) = \pi_1(\mathbb{CP}^2 - \bar{C} \cup H)$ is isomorphic to the fundamental group of the trefoil knot, and has Alexander polynomial $t^2 - t + 1$ (see [10], §7). By Remark 3.12, the higher-order degrees of C are the same as Cochran-Harvey higher-order degrees for the trefoil knot, i.e. $\delta_0(C) = 2$, and $\delta_n(C) = 1$ for all $n > 0$.

(b) Let $\bar{C} \subset \mathbb{CP}^2$ be a curve of degree 6 with 6 cusps as its only singular points, but this time we assume that the six cusps are not on a conic. Then $\pi_1(\mathbb{CP}^2 - \bar{C})$ is abelian, (isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$). Assuming the line H as above is generic and setting $C = \bar{C} - H$, this implies that $\pi_1(\mathbb{C}^2 - C)$ is abelian as well. Therefore, $\delta_n(C) = 0$ for all $n \geq 0$.

From (a) and (b) we see that the higher-order degrees of a curve, at any level n, are also sensitive to the position of singular points. An interesting open problem is to find Zariski pairs that are distinguished by some δ_k, but not distinguished by any δ_n for $n < k$.

Example 5.7. *Oka's curves.*

M. Oka [22] has constructed the curves $\bar{C}_{p,q} \subset \mathbb{CP}^2$ (p, q - relatively prime), with pq singular points locally defined by

$$x^p + y^q = 0,$$

such that $\pi_1(\mathbb{CP}^2 - \bar{C}_{p,q}) = \mathbb{Z}_p * \mathbb{Z}_q$. In fact, the curve $\bar{C}_{p,q}$ is defined by the equation:

$$(x^p + y^p)^q + (y^q + z^q)^p = 0.$$

Fix a generic line $H \subset \mathbb{CP}^2$, and set $C_{p,q} = \bar{C}_{p,q} - H$. Then $\pi_1(\mathbb{C}^2 - C_{p,q}) = \pi_1(\mathbb{CP}^2 - \bar{C}_{p,q} \cup H)$ is isomorphic to the fundamental group of the torus knot of type (p, q). The associated Alexander polynomial is (see for instance [10], §7):

$$\Delta(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

By Remark 3.12 and Proposition 3.10, we obtain: $\delta_0(C_{p,q}) = \deg\Delta(t) = (p-1)(q-1)$, and $\delta_n(C_{p,q}) = pq - p - q$ for all $n > 0$.

Example 5.8. *Branching curves of generic projections. Braid groups.*

Let V_k be a degree k non-singular surface in $\mathbb{CP}^3$ and $\alpha : V_k \to \mathbb{CP}^2$ be a generic projection. If $\bar{C}_k \subset \mathbb{CP}^2$ denotes the branching locus of α, then $\bar{C}_k$ is an irreducible curve of degree $k(k-1)$ with $k(k-1)(k-2)(k-3)/2$ nodes and $k(k-1)(k-2)$ cusps. In the case $k = 3$, one obtains as branching locus the six-cuspidal sextic with all cusps on a conic.

If C_k is the affine curve obtained from $\bar{C}_k$ by removing the intersection with a generic line, then Moishezon [20] showed that $\pi_1(\mathbb{C}^2 - C_k)$ is Artin's braid group on k strands, B_k. The Reidemeister-Schreier process [18] leads to the explicit computation of B'_k/B''_k. For $k \geq 5$, $B'_k/B''_k = 0$, hence C_k has a trivial Alexander polynomial. By Proposition 5.1 we obtain that $\delta_n(C_k) = 0$, for all $n \geq 0$. For $k = 3$, B_3 is the fundamental group of the trefoil knot, so by Example 5.6(a) we obtain: $\delta_0(C_3) = 2$ and $\delta_n(C_3) = 1$ for all $n > 0$.

The case $k = 4$ requires more work. Here we will only calculate δ_0 and δ_1 of the corresponding curve C_4. The Alexander polynomial of C_4 is $t^2 - t + 1$ (see for example [13]), thus

$\delta_0(C_4) = 2$. A presentation for the braid group on four strands is:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$

By using Reidemeister-Schreier techniques, we can obtain a presentation for B_4'. (This was calculated in [7].)

$$B_4' = \langle p, q, a, b, c | pap^{-1} = b, pbp^{-1} = b^2c, qaq^{-1} = c, qbq^{-1} = c^3a^{-1}c, c = a^{-1}b \rangle,$$

where $p = \sigma_2\sigma_1^{-1}$, $q = \sigma_1\sigma_2\sigma_1^{-2}$, $a = \sigma_3\sigma_1^{-1}$, $b = \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}$, and $c = \sigma_1\sigma_2\sigma_1^{-2}\sigma_3\sigma_1\sigma_2^{-1}\sigma_1^{-1}$. Then, $B_4'/B_4'' \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by p and q. Notice that since B_4'/B_4'' is torsion-free, $(B_4)_r'' = B_4''$. Hence by (3.1), we have:

$$\delta_1 = \mathrm{rk}_{\mathbb{K}_1} \left(B_4''/B_4''' \otimes_{\mathbb{Z}\bar{\Gamma}_1} \mathbb{K}_1 \right),$$

where $\bar{\Gamma}_1 = \ker(\bar{\psi} : B_4/B_4'' \to B_4/B_4') = B_4'/B_4''$. Therefore, we must understand B_4''/B_4''' as a $\mathbb{Z}[p^{\pm 1}, q^{\pm 1}]$-module, and then determine the rank as a $\mathbb{Q}(p, q)$-vector space.

Again using Reidemeister-Schreier techniques, we calculate a group presentation for B_4'':

$$\begin{aligned} B_4'' = \langle \rho_{i,j}, \alpha_{i,j} | \rho_{i,0} = 1, \rho_{i,j}\alpha_{i+1,j}\rho_{i,j}^{-1} = \alpha_{i,j}\alpha_{i,j+1}, \rho_{i,j}\alpha_{i+1,j+1}\rho_{i,j}^{-1} = \alpha_{i,j}\left(\alpha_{i,j+1}\right)^2, \\ \alpha_{i,j+2} = \left(\alpha_{i,j+1}\right)^2 \alpha_{i,j}^{-1}\alpha_{i,j+1} \rangle, \end{aligned}$$

where $\rho_{i,j} = p^i q^j p q^{-j} p^{-(i+1)}$ and $\alpha_{i,j} = p^i q^j a q^{-j} p^{-i}$. Notice that p and q act on B_4'' by conjugation. Furthermore, $p * (q * \gamma) = \rho_{0,1}^{-1}(q * (p * \gamma))\rho_{0,1}$, for all $\gamma \in B_4''$. Hence although the actions of p and q do not commute in B_4'', they do commute in B_4''/B_4'''. In particular, B_4''/B_4''' is indeed a $\mathbb{Z}[p^{\pm 1}, q^{\pm 1}]$-module.

We have the following presentation for B_4''/B_4''' as an abelian group:

$$B_4''/B_4''' = \langle \rho_{i,j}, \alpha_{i,j} | \rho_{i,0} = 0, \alpha_{i+1,j} = \alpha_{i,j} + \alpha_{i,j+1}, \alpha_{i+1,j+1} = \alpha_{i,j} + 2\alpha_{i,j+1}, \alpha_{i,j+2} = 3\alpha_{i,j+1} - \alpha_{i,j} \rangle.$$

To get a presentation as a $\mathbb{Z}[p^{\pm 1}, q^{\pm 1}]$-module, we note that:

$$\begin{aligned} \rho_{i,j} &= \prod_{k=1}^{j} (p^i q^{j-k} * \rho_{0,1}), \text{ for } j \geq 1, \\ \rho_{i,j} &= \prod_{k=1}^{-j} (p^i q^{j-1+k} * \rho_{0,1}^{-1}), \text{ for } j \leq -1, \\ \rho_{i,0} &= 0, \\ \alpha_{i,j} &= p^i q^j * \alpha_{0,0}, \text{ for all } i, j \in \mathbb{Z}. \end{aligned}$$

Therefore, as a $\mathbb{Z}[p^{\pm 1}, q^{\pm 1}]$-module, B_4''/B_4''' is generated by $\rho_{0,1}$ and $\alpha_{0,0}$. Furthermore, $\rho_{0,1}$ generates a free submodule, while $\alpha_{0,0}$ generates a torsion submodule. Hence the rank as a $\mathbb{Q}(p, q)$-vector space is 1. Therefore, $\delta_1(C_4) = 1$.

Note. The background material on the constructions mentioned in this example are beautifully explained in Libgober's papers [13] and [14]. In particular, the latter contains a summary of Moishezon's results [20].

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