LINK CONCORDANCE AND HIGHER-ORDER BLANCHFIELD DUALITY

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Abstract. We introduce a new technique for showing classical knots and links are not slice. As one application we show that the iterated Bing doubles of many algebraically slice knots are not topologically slice. Some of the proofs do not use the existence of the Cheeger-Gromov bound, a deep analytical tool used by Cochran-Teichner. Our main examples are actually boundary links but cannot be detected in the algebraic boundary link concordance group, nor by any ρ invariants associated to solvable representations into finite unitary groups.

1. Introduction

A *link* $L = \{K_1, ..., K_m\}$ of m-components is an ordered collection of m oriented circles disjointly embedded in S^3. A *knot* is a link of one component. A *topologically slice link* (abbreviated as *slice* in this paper) is a link whose components bound a disjoint union of m 2-disks topologically and locally flatly embedded in B^4. The question of which links are slice links lies at the heart of the topological classification of 4-dimensional manifolds.

The connected sum operation gives the set of all knots, modulo slice knots, the structure of an abelian group, called the **topological knot concordance group** $\mathcal{C}$, which is a quotient of its smooth analogue. For links one must consider *string* links to get a well-defined group structure, and this operation is not commutative [27]. This group is called the m-component **string link concordance group**. We applied our techniques to knot concordance in [10][12]. This paper gives new information about link concordance. All of the results here were announced in [12] and appeared first in [11]. We employ the Cheeger-Gromov von Neumann ρ-invariants and higher-order Alexander modules that were introduced in [14]. Our new technique is to expand upon previous results of Leidy concerning higher-order Blanchfield forms *without localizing the coefficient system* [24] [23]. This is used to show that certain elements of π_1 of a slice knot (or link) exterior cannot lie in the kernel of the map into any slice disk(s) exterior. We also employ results of Harvey on the *torsion-free derived series of groups* [21], and recent results of Cochran-Harvey on versions of Dwyer's Theorem for the derived series [18]. We note that the construction of examples is in the smooth category so that we actually also prove the corresponding statements about smooth link concordance.

Natural families of links have been considered. In particular, if K is any knot then the *Bing-double of* K, $BD(K)$ is the 2-component link shown in Figure 1.1. If K is slice then it is easy to see that $BD(K)$ is a slice link. A natural question is whether or not the converse is true. It was shown by Harvey that if the Bing double (or even an iterated Bing-double) of K is topologically slice then the integral over the circle of the Levine signatures of K is zero [21, Corollary 5.6]. It was shown by Cimasoni that if $BD(K)$ is a *boundary slice* link then K is

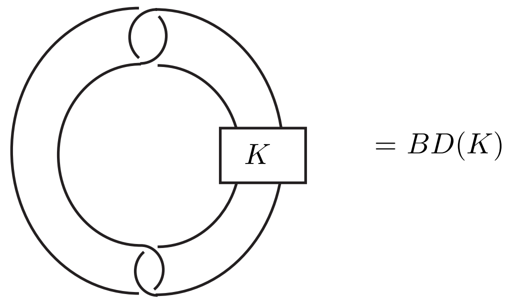


FIGURE 1.1. Bing double of K

algebraically slice [8]. Subsequently it was shown by Cha-Livingston-Ruberman that if $BD(K)$ is a slice link then K must be an algebraically slice knot [6]. Here we address the questions: If K is algebraically slice then does it follow that $BD(K)$ is a topological slice link? What about for iterated Bing doubles? We answer these questions in the negative by showing that certain higher-order signatures of K offer further obstructions. For example, in Section 4 we define **first-order signatures** of K, akin to Casson-Gordon invariants, and show that the first-order signatures of K, like the ordinary signatures, obstruct any **iterated Bing double** of K from being a slice link. This improves on Harvey's theorem.

Theorem 4.4. *Let K be an arbitrary knot. If some iterated Bing double of K is topologically slice in a rational homology 4-ball then one of the first-order signatures of K is zero.*

For example, for the algebraically slice knots J_1 of Figure 1.2, the first order signatures of J_1 are related to classical signatures of J_0, and similarly for the knots E_1 as in Figure 1.3, which are of order 2 in the algebraic concordance group.

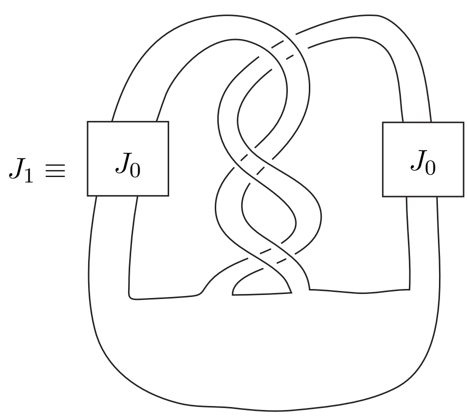


FIGURE 1.2. Algebraically Slice Knots J_1

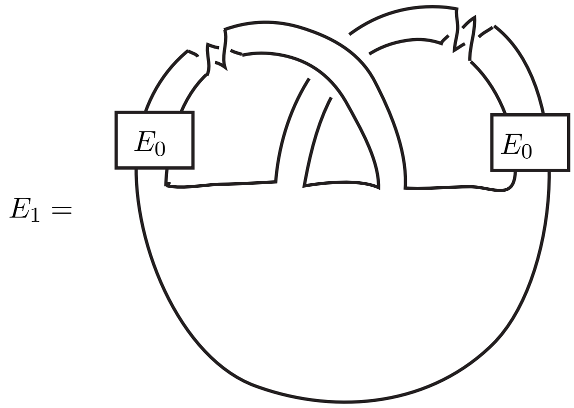


FIGURE 1.3. E_1

For a knot K in S^3 let $\rho_0(K)$ denote the integral over the circle of the classical Levine signature function of K (normalize so that the length of the circle is 1).

On these examples, Theorem 4.4 takes the following nice form:

Corollary 4.5. *If J_1 is the algebraically slice knot of Figure 1.2 then there is a constant C such that if some iterated Bing double of J_1 is slice in a rational homology ball then $\rho_0(J_0) \in \{0, C\}$. If E_1 is a knot as in Figure 1.3 then if some iterated Bing double of E_1 is slice in a rational homology ball then $\rho_0(E_0) = 0$.*

We remark that subsequent work of Cha shows that even many amphichiral knots have non-slice Bing doubles [5]. Amphichiral knots cannot be handled by the present paper.

We have similar results for iterated Bing doubles of the even more subtle knots of the (recursively-defined) family J_n, $n > 0$, of Figure 1.4 which are not only algebraically slice but also have vanishing Casson-Gordon invariants for every $n > 1$. An n^{th}-order higher-order signature of J_n obstructs the iterated Bing doubles of J_n from being slice links. Moreover these iterated Bing doubles give non-trivial examples of links that lie deeper and deeper in the Cochran-Orr-Teichner filtration of the set of concordance classes of links

$$\cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}.$$

defined in [14, Sections 7,8] and reviewed in our Section 6. Recall that a link in $\mathcal{F}_n$ is called $(\mathbf{n}) - \mathbf{solvable}$.

Corollary 5.3. *For any knot J_0 with invariant zero and any n, there is a constant C such that if $|\rho_0(J_0)| > C$ then the Bing double of J_{n-1} is (n)-solvable but not slice nor even $(n+1)$-solvable.*

Corollary 5.4. *Suppose k and n are positive integers, and J_0 is a knot with Arf$(J_0) = 0$. Then there is a constant C such that if $|\rho_0(J_0)| > C$, then the k-fold iterated Bing double of J_{n-k} is (n)-solvable but not slice nor even $(n+1)$-solvable.*

The specific families of links of Figure 1.1 are important because of their simplicity. However, they are merely particular instances of a more general "doubling" phenomenon to which

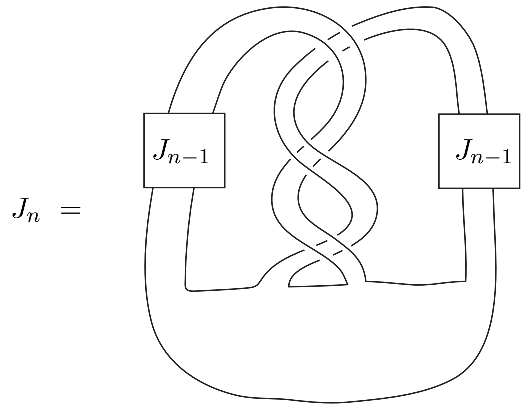


FIGURE 1.4. The recursive family $J_{n+1}, n \geq 0$

our techniques may be applied. In order to state these results, we review a method we will use to construct examples. Let R be a link in S^3 and $\{\eta_1, \eta_2, \ldots, \eta_m\}$ be an oriented trivial link in S^3 which misses R bounding a collection of disks that meet R transversely. Suppose $\{K_1, K_2, \ldots, K_m\}$ is an m-tuple of auxiliary knots. Let $R(\eta_1, \ldots, \eta_m\, K_1, \ldots, K_m)$ denote the result of the operation pictured in Figure 1.5, that is, for each η_i, take the embedded disk in S^3 bounded by η_i; cut off R along the disk; grab the cut strands, tie them into the knot K_i (with no twisting) and reglue as shown in Figure 1.5.

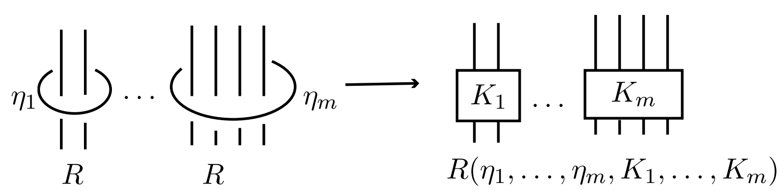


FIGURE 1.5. $R(\eta_1, \ldots, \eta_m, K_1, \ldots, K_m)$: Infection of R by K_i along η_i

We will call this the result of *infection performed on the link R using the infection knots K_i along the curves η_i*. This construction can also be described in the following way. For each η_i, remove a tubular neighborhood of η_i in S^3 and glue in the exterior of a tubular neighborhood of K_i along their common boundary, which is a torus, in such a way that the longitude of η_i is identified with the meridian of K_i and the meridian of η_i with the reverse of the longitude of K_i. The resulting space can be seen to be homeomorphic to S^3 and the image of R is the new link. In the case that $m = 1$ this is the same as the classical satellite construction. In general it can be considered to be a 'generalized satellite construction', widely utilized in the study of knot concordance. In the case that $m = 1$ and $lk(\eta, R) = 0$ it is precisely the same as forming a satellite of J with winding number zero. This yields an operator

$$R_\eta : \mathcal{C} \to \mathcal{C}^k.$$

where $\mathcal{C}^k$ is the set of concordance classes of k-component links. For general m with $lk(\eta_i, R) = 0$, it should be considered as a *generalized doubling operator*, R_{η_i}, parameterized by $(R, \{\eta_i\})$. If, for simplicity, we assume that all "input knots" assume identical then such an operator is a function

$$R_{\eta_i} : \mathcal{C} \to \mathcal{C}^k.$$

Bing-doubling is an example of this $(m = 1)$ as suggested by Figure 1.6.

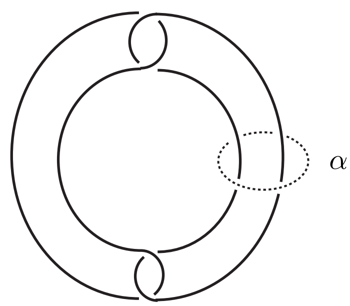


FIGURE 1.6. Bing double of K is infection on the trivial link along α using K

Another primary example is the "9_{46}-doubling" operation of going from the left-hand side of Figure 1.7 to the right-hand side. Here R is the 9_{46} knot and $\{\eta_1, \eta_2\} = \{\alpha, \beta\}$ are as shown on the left-hand side of Figure 1.7. The image of a knot K under the operator $R_{\alpha,\beta}$ is denoted by $R(K)$ and is shown on the right-hand side of Figure 1.7. Note that our previously defined knot J_1 is the same as $R(J_0)$.

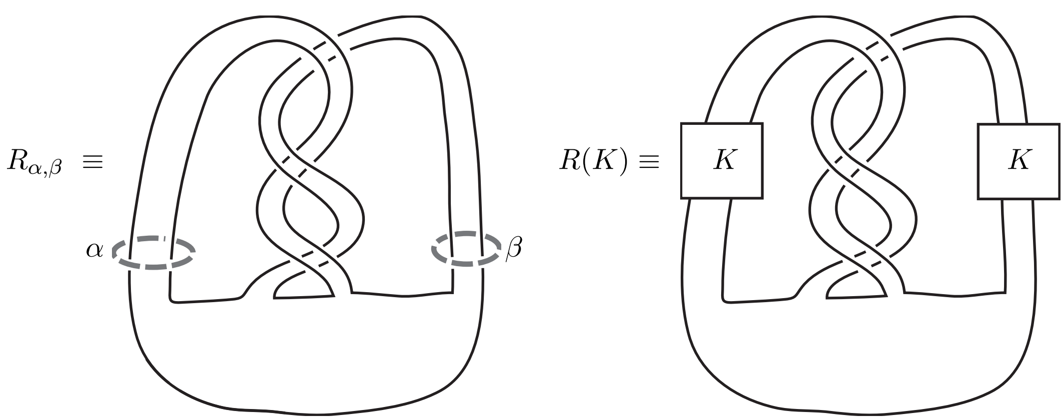


FIGURE 1.7. R-doubling

Most of the results of this paper concern to what extent these functions are injective. The point is that, because of the condition on "winding numbers", $lk(\eta_i, R) = 0$, if R is a slice link, the images of such operators R contain only links for which the classical Seifert-matrix-type invariants vanish. Moreover these operators respect the COT filtration.

Proposition 1.1. [15, proof of Proposition 3.1]*(see also Lemma 6.4 of this paper) If R is a slice link and $\eta_i \in \pi_1(S^3 - R)^{(n)}$ then the operator R_{η_i} satisfies*

$$R_{\eta_i}(\mathcal{F}_0) \subset \mathcal{F}_n.$$

Thus *iterations* of these operators, *iterated generalized doubling*, produce increasingly subtle links. More generally let us define an *n-times iterated generalized doubling* to be precisely such a composition of operators using possibly different slice links R_j, and different curves $\eta_{j1}, \ldots, \eta_{jm_j}$. For example the knot J_n of Figure 1.4 is obtained from J_0 by applying $R \circ \cdots \circ R$ where $R = R_{\alpha,\beta}$ is as in Figure 1.7. Then, generalizing Corollary 5.4, our method establishes:

Theorem 5.16. *Suppose T is a slice link, α is an unknotted circle in $S^3 - T$ that represents an element in $\pi_1(S^3 - T)^{(k)}$ but not in $\pi_1(M_T)^{(k+1)}_H$. Suppose for each j, $1 \le j \le n-k$, R_j is a slice knot, $\{\eta_{j1}, \ldots, \eta_{jm_j}\}$ is a trivial link of circles in $S^3 - R_j$ with the property that the submodule of the classical Alexander polynomial of R_j generated by $\{\eta_{j1}, \ldots, \eta_{jm_j}\}$ contains elements x, y such that $\mathcal{B}\ell_0^j(x,y) \neq 0$, where $\mathcal{B}\ell_0^j$ is the Blanchfield form of R_j. Finally suppose that Arf(K)= 0 Then the result, $L(K) \equiv T_\alpha \circ R_{n-k} \circ \cdots \circ R_1(K)$, of the iterated generalized doubling (applied to K) lies in $\mathcal{F}_n$ and there is a constant C, such that if $|\rho_0(K)| > C$, then $L(K)$ is of infinite order in the topological concordance group (moreover no multiple lies in $\mathcal{F}_{n+1}$).*

2. Higher-Order Signatures and How to Calculate Them

In this section we review the von Neumann ρ-invariants and explain to what extent they are concordance invariants. We also show how to calculate them for knots or links that are obtained from the infections defined in Section 1.

The use of variations of Hirzebruch-Atiyah-Singer signature defects associated to covering spaces is a theme common to most of the work in the field of knot and link concordance since the 1970's. In particular, Casson and Gordon initiated their use in cyclic covers [1] [2]; Farber, Levine and Letsche initiated the use of signature defects associated to general (finite) unitary representations [26] [25]; and Cochran-Orr-Teichner initiated the use of signatures associated to the left regular representations [14]. See [19] for a beautiful comparison of these approaches in the metabelian case.

Given a compact, oriented 3-manifold M, a discrete group Γ, and a representation $\phi : \pi_1(M) \to \Gamma$, the *von Neumann ρ-invariant* was defined by Cheeger and Gromov by choosing a Riemannian metric and using η-invariants associated to M and its covering space induced by ϕ. It can be thought of as an oriented homeomorphism invariant associated to an arbitrary regular covering space of M [7]. If $(M,\phi) = \partial(W,\psi)$ for some compact, oriented 4-manifold W and $\psi : \pi_1(W) \to \Gamma$, then it is known that $\rho(M,\phi) = \sigma^{(2)}_\Gamma(W,\psi) - \sigma(W)$ where $\sigma^{(2)}_\Gamma(W,\psi)$ is the $L^{(2)}$-signature (von Neumann signature) of the intersection form defined on $H_2(W;\mathbb{Z}\Gamma)$ twisted by ψ and $\sigma(W)$ is the ordinary signature of W [30]. In the case that Γ is a poly-(torsion-free-abelian) group (abbreviated **PTFA group** throughout), it follows that $\mathbb{Z}\Gamma$ is a right Ore domain that embeds into its (skew) quotient field of fractions $\mathcal{K}\Gamma$ [31, pp.591-592, Lemma 3.6ii p.611]. In this case $\sigma^{(2)}_\Gamma$ is a function of the Witt class of the equivariant intersection form on $H_2(W;\mathcal{K}\Gamma)$ [14, Section 5]. In the special case (such as $\beta_1(M) = 1$) that this form is non-singular, it can be thought of as a homomorphism from $L^0(\mathcal{K}\Gamma)$ to $\mathbb{R}$.

All of the coefficient systems Γ in this paper will be of the form $\pi/\pi_r^{(n)}$ where π is the fundamental group of a space (usually a 4-manifold) and $\pi_r^{(n)}$ is the n^{th}-term of the **rational derived series**. The latter was first considered systematically by Harvey. It is defined by

$$\pi_r^{(0)} \equiv \pi,\ \pi_r^{(n+1)} \equiv \{x \in \pi_r^{(n)} | \exists k \neq 0, x^k \in [\pi_r^{(n)}, \pi_r^{(n)}]\}.$$

Note that n^{th}-term of the usual derived series $\pi^{(n)}$ is contained in the n^{th}-term of the rational derived series. For free groups and knot groups, they coincide. It was shown in [22, Section 3] that $\pi/\pi_r^{(n)}$ is a PTFA group.

The utility of the von Neumann signatures lies in the fact that they obstruct knots from being slice knots. It was shown in [14, Theorem 4.2] that, under certain situations, higher-order von Neumann signatures vanish for slice knots, generalizing the classical result of Murasugi and the results of Casson-Gordon. That proof fails for links, but the extension was later accomplished by Harvey (there is an extra obstruction). Moreover, Cochran-Orr-Teichner defined a filtration on knots and links and showed that certain higher-order signatures obstructed a knot's lying in a certain term of the filtration. Harvey also extended this to links. Here we state the needed results for slice knots and links. In an Section 6 we review the filtration and the more general results. In the case of links we prove a more general result than Harvey's, which will be needed later.

First,

Theorem 2.1. *(Cochran-Orr-Teichner* [14, Theorem 4.2]*) If a knot K is topologically slice in a rational homology 4-ball and $\phi : \pi_1(M_K) \to \Gamma$ is a PTFA coefficient system that extends to the fundamental group of the exterior of the slicing disk, then $\rho(M_K, \phi) = 0$.*

The analogous result for links has not specifically appeared, although it is implicit in and follows from the techniques of [21]. The proof will be given as a corollary of a more general in Section 6).

Theorem 2.2. *(Corollary of Theorem 6.7) If a link L is topologically slice in a rational homology 4-ball and $\phi : \pi_1(M_L) \to \Gamma$ is a PTFA coefficient system that extends to the fundamental group of the exterior of the slicing disks, then $\rho(M_L, \phi) = 0$.*

Some other useful properties of von Neumann ρ-invariants are given below. One can find detailed explanations of most of these in [14, Section 5]. The last property, that for a fixed 3-manifold, the set $\{\rho(M, \phi)\}$ is bounded above and below, is an analytical result of Cheeger and Gromov that we use in some (but not all) of our results here.

Proposition 2.3. *Let M be a closed,oriented 3-manifold and $\phi : \pi_1(M) \to \Gamma$ as above.*

(1) *If $(M, \phi) = \partial(W, \psi)$ for some compact oriented 4-manifold W such that the equivariant intersection form on $H_2(W;\mathcal{K}\Gamma)/j_*(H_2(\partial W;\mathcal{K}\Gamma))$ admits a half-rank summand on*

which the form vanishes, then $\sigma^{(2)}_\Gamma(W,\psi) = 0$ *(see* [21, Lemma 3.1 and Remark 3.2] *for a proper explanation of this for manifolds with* $\beta_1 > 1$*). Thus if* $\sigma(W) = 0$ *then* $\rho(M,\phi) = 0$

(2) *If* ϕ *factors through* $\phi' : \pi_1(M) \to \Gamma'$ *where* Γ' *is a subgroup of* Γ*, then* $\rho(M,\phi') = \rho(M,\phi)$*.*

(3) *If* ϕ *is trivial (the zero map), then* $\rho(M,\phi) = 0$*.*

(4) *If* $M = M_K$ *is zero surgery on a knot* K *and* $\phi : \pi_1(M) \to \mathbb{Z}$ *is the abelianization, then* $\rho(M,\phi)$ *is equal to the integral over the circle of the Levine (classical) signature function of* K*, normalized so that the length of the circle is* 1 [15, Prop. 5.1]*.* ***This real number will be denoted*** $\rho_0(K)$*.*

(5) *(Cheeger-Gromov* [7]*) Given* M*, there is a positive constant* C_M*, the* ***Cheeger-Gromov constant of M****, such that for every* ϕ

$$|\rho(M,\phi)| < C_M.$$

The following elementary lemma reveals the additivity of the ρ-invariant under infection. It is only slightly more general than [15, Proposition 3.2]. The use of a Mayer-Vietoris sequence to analyze the effect of a satellite construction on signature defects is common to essentially all of the previous work in this field (see for example [28]).

Suppose $L = R(\eta_i, K_i)$ is obtained by infection as described in Section 1. Let the zero surgeries on R, L, and K_i be denoted M_R M_L, M_i respectively. Suppose $\phi : \pi_1(M_L) \to \Gamma$ is a map to an arbitrary PTFA group Γ such that, for each i, ℓ_i, the longitude of K_i, lies in the kernel of ϕ. Since $S^3 - K_i$ is a submanifold of M_L, ϕ induces a map on $\pi_1(S^3 - K_i)$. Since l_i lies in the kernel of ϕ this map extends uniquely to a map that we call ϕ_i on $\pi_1(M_i)$. Similarly, ϕ induces a map on $\pi_1(M_R - \coprod \eta_i)$. Since M_R is obtained from $(M_R - \coprod \eta_i)$ by adding m 2-cells along the meridians of the η_i, $\mu(\eta_i)$ and m 3−cells, and since $\mu(\eta_i) = l_i^{-1}$ and $\phi_i(l_i) = 1$, ϕ extends uniquely to ϕ_R. Thus ϕ induces unique maps ϕ_i and ϕ_R on $\pi_1(M_i)$ and $\pi_1(M_R)$ (characterized by the fact that they agree with ϕ on $\pi_1(S^3 - K_i)$ and $\pi_1(M_R - \coprod \eta_i)$ respectively).

There is a very important case when the hypothesis above that $\phi(\ell_i) = 1$ is always satisfied. Namely suppose $\Gamma^{(n+1)} = 1$ and $\eta_i \in \pi_1(M_R)^{(n)}$. Since a longitudinal push-off of η_i, called ℓ_{η_i} or η_i^+, is isotopic to η_i in the solid torus $\eta_i \times D^2 \subset M_R$, $\ell_{\eta_i} \in \pi_1(M_R)^{(n)}$ as well. By [9, Theorem 8.1] or [23] it follows that $\ell_{\eta_i} \in \pi_1(M_L)^{(n)}$. Since μ_i, the meridian of K_i, is identified to ℓ_{η_i}, $\mu_i \in \pi_1(M_L)^{(n)}$ so $\phi(\mu_i) \in \Gamma^{(n))}$ for each i. Thus $\phi_i(\pi_1(S^3 - K_i)^{(1)}) \subset \Gamma^{(n+1)} = \{e\}$ and in particular the longitude of each K_i lies in the kernel of ϕ.

Lemma 2.4. [10, Lemma 2.3] *In the notation of the two previous paragraphs (assuming* $\phi(\ell_i) = 0$ *for all* i*),*

$$\rho(M_L,\phi) - \rho(M_R,\phi_R) = \sum_{i=1}^{m} \rho(M_i,\phi_i).$$

Moreover if $\pi_1(S^3 - K_i)^{(1)} \subset$ *kernel(*ϕ_i*) then either* $\rho(M_i,\phi_i) = \rho_0(K_i)$*, or* $\rho(M_i,\phi_i) = 0$*, according as* $\phi_R(\eta_i) \neq 1$ *or* $\phi_R(\eta_i) = 1$*. Specifically, if* $\Gamma^{(n+1)} = 1$ *and* $\eta_i \in \pi_1(M_R)^{(n)}$ *then this is the case.*

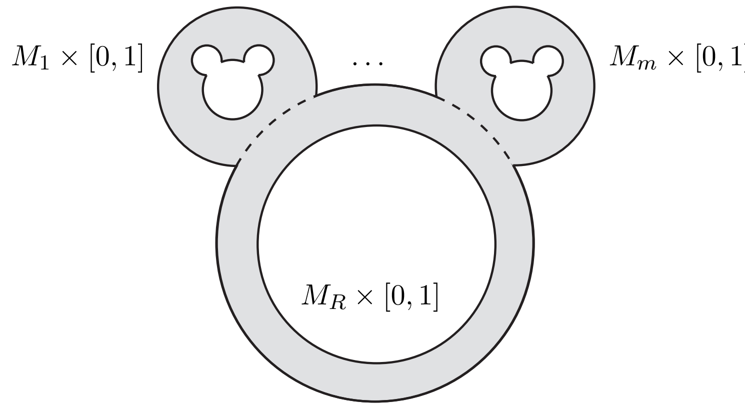


FIGURE 2.1. The cobordism E

The dashed arcs in the figure represent the solid tori $\eta_i \times D^2$. Observe that the 'outer' boundary component of E is M_L. Note that E deformation retracts to $\overline{E} = M_L \cup (\coprod_i (\eta_i \times D^2))$, where each solid torus is attached to M_L along its boundary. Hence $\overline{E}$ is obtained from M_L by adding m 2-cells along the loops $\mu(\eta_i) = l_i$, and m 3-cells. Thus, by our assumption, ϕ extends uniquely to $\overline{\phi} : \pi_1(\overline{E}) \to \Gamma$ and hence $\overline{\phi} : \pi_1(E) \to \Gamma$. Clearly the restrictions of $\overline{\phi}$ to $\pi_1(M_i)$ and $\pi_1(M_R \times \{0\})$ agree with ϕ_i and ϕ_R respectively. It follows that that

$$\rho(M_L, \phi) - \rho(M_R, \phi_R) = \sum_{i=1}^{m} \rho(M_i, \phi_i) + \sigma^{(2)}(E, \overline{\phi}) - \sigma(E).$$

Lemma 2.5. [10, Lemma 2.4] *With respect to* any *coefficient system, $\phi : \pi_1(E) \to \Gamma$, the signature of the equivariant intersection form on the $H_2(E; \mathbb{Z}\Gamma)$ is zero.*

We want to collect, in the form of a Lemma, the properties of the cobordism E that we have established in the proofs above. These will be used in later sections.

Lemma 2.6. [10, Lemma 2.5] *With regard to E as above, the inclusion maps induce*

(1) *an epimorphism $\pi_1(M_L) \to \pi_1(E)$ whose kernel is the normal closure of the longitudes of the infecting knots K_i viewed as curves $\ell_i \subset S^3 - K_i \subset M_L$;*
(2) *isomorphisms $H_1(M_L) \to H_1(E)$ and $H_1(M_R) \to H_1(E)$;*
(3) *and isomorphisms $H_2(E) \cong H_2(M_L) \oplus_i H_2(M_{K_i}) \cong H_2(M_R) \oplus_i H_2(M_{K_i})$.*
(4) *The longitudinal push-off of η_i, $\ell_{\eta_i} \subset M_L$ is isotopic in E to $\eta_i \subset M_R$ and to the meridian of K_i , $\mu_i \subset M_{K_i}$.*
(5) *The longitude of K_i, $\ell_i \subset M_{K_i}$ is isotopic in E to the reverse of the meridian of η_i, $\eta_i^{-1} \subset M_L$ and to the longitude of K_i in $S^3 - K_i \subset M_L$ and to the reverse of the meridian of η_i, $(\mu_{\eta_i})^{-1} \subset M_R$ (the latter bounds a disk in M_R).*

3. Higher-Order Blanchfield forms for knots and links

We have seen in Lemma 2.4 that an infection will have an effect on a ρ-invariant only if the infection circle η survives under the map defining the coefficient system. Therefore it is important to prove *injectivity* theorems concerning $\pi_1(S^3-R)\to\pi_1(B^4-\Delta)$, that is, loosely speaking, to prove that η survives under the map

$$j_*:\pi_1(S^3-R)^{(n)}/\pi_1(S^3-R)^{(n+1)}\to\pi_1(B^4-\Delta)^{(n)}/\pi_1(B^4-\Delta)^{(n+1)}.$$

Higher-order Alexander modules are relevant to this task since the latter quotient can be interpreted as $H_1(W_n)$ where W_n is the (solvable) covering space of $B^4-\Delta$ corresponding to the subgroup $\pi_1(B^4-\Delta)^{(n)}$. Such modules were named *higher-order Alexander modules* in [14] [9] [22]. We will employ higher-order Blanchfield linking forms on higher-order Alexander modules to find restrictions on the kernels of such maps. The logic of the technique is entirely analogous to the classical case ($n=1$): Any two curves η_0,η_1, say, that lie in the kernel of j_* must satisfy $\mathcal{B}\ell(\eta_0,\eta_0)=\mathcal{B}\ell(\eta_0,\eta_1)=\mathcal{B}\ell(\eta_1,\eta_1)=0$ with respect to a higher order linking form $\mathcal{B}\ell$. Our major new insight is that, if the curves lie in a submanifold $S^3-K\hookrightarrow S^3-J$, a situation that arises whenever J is formed from R by infection using a knot K, then the values (above) of the higher-order Blanchfield form of J can be expressed in terms of the values of the classical Blanchfield form of K!

Higher-order Alexander modules and higher-order linking forms for classical knot exteriors and for closed 3-manifolds with $\beta_1(M)=1$ were introduced in [14, Theorem 2.13] and further developed in [9] and [24]. These were defined on the so called higher-order Alexander modules. Higher-order Alexander modules for *links and* 3*-manifolds* in general were defined and investigated in [22]. Blanchfield forms for 3-manifolds with $\beta_1(M)>1$ were only recently defined by Leidy [23]. It is crucial to our techniques that we work with such Blanchfield forms without localizing the coefficient systems, as was investigated in [24] [23]. It is in this aspect that our work deviates from that of [14] [15] [13]. A non-localized Blanchfield form for knots also played a crucial role in [20].

First we recall that *higher-order Blanchfield linking forms* have been defined under fairly general circumstances.

Theorem 3.1. *[[23, Theorem 2.3]] Suppose M is a closed, connected, oriented* 3*-manifold and $\phi:\pi_1(M)\to\Lambda$ is a PTFA coefficient system. Suppose $\mathcal{R}$ is a classical Ore localization of the Ore domain $\mathbb{Z}\Lambda$ (so $\mathbb{Z}\Lambda\subset\mathcal{R}\subset\mathcal{K}\Lambda$). Then there is a linking form:*

$$\mathcal{B}l^M_{\mathcal{R}}:TH_1(M;\mathcal{R})\to(TH_1(M;\mathcal{R}))^{\#}\equiv\overline{Hom_{\mathcal{R}}(TH_1(M;\mathcal{R}),\mathcal{K}\Lambda/\mathcal{R})}.$$

An *Ore localization* of $\mathbb{Z}\Lambda$ is $\mathcal{R}=\mathbb{Z}\Lambda[S^{-1}]$ for some right-Ore set S [32]. When we speak of the *unlocalized* Blanchfield form we mean that $\mathcal{R}=\mathbb{Z}\Lambda$ or $\mathcal{R}=\mathbb{Q}\Lambda$. $TH_1(M;\mathcal{R})$ denotes the $\mathcal{R}$-torsion submodule. In general $TH_1(M;\mathcal{R})$ need not have homological dimension one nor even be finitely-generated, and these linking forms are *singular*.

Leidy analyzed the effect of an infection on the unlocalized Blanchfield forms in [24][23]. This generalizes the result on the classical Blanchfield form for satellite knots [29]. If L is obtained by infection on a link R along a circle α using the knot K and $\phi:\pi_1(M_L)\to\Lambda$ is a PTFA coefficient system, and $\mathbb{Z}\Lambda\subset\mathcal{R}\subset\mathcal{K}\Lambda$ then $\mathcal{B}l^L_{\mathcal{R}}$ is defined. On the other hand, by definition, exterior of the knot K is a submanifold of M_L and there is an induced coefficient system, that we also call ϕ, with respect to which there is a Blanchfield linking form (first defined in [14, Theorem 2.13])

$$\mathcal{B}l^K_{\mathcal{R}} : TH_1(S^3 - K; \mathcal{R}) \to (TH_1(S^3 - K; \mathcal{R}))^{\#}.$$

(We note that if ϕ is nontrivial when restricted to $\pi_1(S^3 - K)$ then $TH_1(S^3 - K; \mathcal{R}) = H_1(S^3 - K; \mathcal{R})$. Otherwise $TH_1(S^3 - K; \mathcal{R}) = 0$ [14, Proposition 2.11]). Then it is an easy exercise for the reader using the geometric definition of these Blanchfield forms (or see [24, Theorem 4.6, proof of property 1]), that these forms are compatible:

Proposition 3.2. [23, Theorem 3.7] *In the situation above the following diagram commutes*

$$\begin{array}{ccc} TH_1(S^3 - K; \mathcal{R}) & \xrightarrow{i_*} & TH_1(M_L; \mathcal{R}) \\ \downarrow \mathcal{B}l^K_{\mathcal{R}} & & \downarrow \mathcal{B}l^{M_L}_{\mathcal{R}} \\ TH_1(S^3 - K; \mathcal{R})^{\#} & \xleftarrow{i^{\#}} & TH_1(M_L; \mathcal{R})^{\#} \end{array} \tag{3.1}$$

that is, for all $x, y \in H_1(S^3 - K; \mathcal{R})$

$$\mathcal{B}l^{M_L}_{\mathcal{R}}(i_*(x), i_*(y)) = \mathcal{B}l^K_{\mathcal{R}}(x, y).$$

Moreover, in some important situations, the induced coefficient system $\phi : \pi_1(S^3 - K) \to \Lambda$ factors through, $\mathbb{Z}$, the abelianization of the knot exterior. In particular if L is obtained by infection on a link R along a circle $\alpha \in \pi_1(M_R)^{(k-1)}$ where $\Lambda^{(k)} = 1$, then this is the case. Furthermore the higher-order Blanchfield form $\mathcal{B}l^K_{\Lambda}$ is merely the classical Blanchfield form on the classical Alexander module, "tensored up". What is meant by this is the following. Supposing that ϕ is both nontrivial and factors through the abelianization, the induced map image$(\phi) \equiv \mathbb{Z} \hookrightarrow \Lambda$ is an embedding so it induces embeddings

$$\phi : \mathbb{Q}[t, t^{-1}] \hookrightarrow \mathbb{Q}\Lambda, \ \ \phi : \mathbb{Q}(t) \hookrightarrow \mathcal{K}\Lambda,$$

and hence an embedding

$$\overline{\phi} : \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}] \hookrightarrow \mathcal{K}\Lambda/\mathbb{Q}\Lambda.$$

Then there is an isomorphism

$$H_1(S^3 \backslash K; \mathbb{Q}\Lambda) \cong H_1(S^3 \backslash K; \mathbb{Q}[t, t^{-1}]) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Lambda \cong \mathcal{A}_0(K) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Lambda,$$

where $\mathcal{A}_0(K)$ is the classical (rational) Alexander module of K and where $\mathbb{Q}\Lambda$ is a $\mathbb{Q}[t, t^{-1}]$-module via the map $t \to \phi(\alpha)$ [9, Theorem 8.2]. Moreover

$$\mathcal{B}l^K_{\Lambda}(x \otimes 1, y \otimes 1) = \overline{\phi}(\mathcal{B}l^K_0(x, y))$$

for any $x, y \in \mathcal{A}_0(K)$, where $\mathcal{B}l^K_0$ is the classical Blanchfield form on the rational Alexander module of K [23, Proposition 3.6] [24, Theorem 4.7] (see also [3, Section 5.2.2]).

Then, finally, Leidy shows that the Blanchfield form on M_L the sum of that on $H_1(M_R)$ and that on the infecting knot K (generalizing the classical result for satellites [29]). We state this below although, in this paper, we shall not need this nontrivial fact that the module $H_1(M_L; \mathbb{Q}\Lambda)$ decomposes, nor even that $\mathcal{A}_0(K) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Lambda$ is a submodule of it. We will only need the almost obvious fact that the inclusion of the 3-manifolds $S^3 - K_i \hookrightarrow M_L$ induces a (natural) map on the Blanchfield forms and that the induced Blanchfield form on $S^3 - K$ is the classical form "tensored up".

Theorem 3.3. *[Theorem 3.7,Proposition 3.4 [23]] Suppose $L = R(\alpha_i, K_i)$ is obtained by infection as above with $\alpha_i \in \pi_1(M_R)^{(k-1)}$ for all i. Let the zero surgeries on R, L, and K_i be denoted M_R M_L, M_i respectively. Suppose Λ is a PTFA group such that $\Lambda^{(k)} = 1$. Suppose $\phi : \pi_1(M_L) \to \Lambda$ is a coefficient system. Then the inclusions induce an isomorphism*

$$H_1(M_R; S^{-1}\mathbb{Z}\Lambda) \oplus_{i\in A} H_1(S^3\backslash K_i; S^{-1}\mathbb{Z}\Lambda) \xrightarrow{i_*} H_1(M_L; S^{-1}\mathbb{Z}\Lambda).$$

where $A = \{i \mid \phi((\alpha_i)^+ \neq 1\}$. Moreover there is an isomorphism

$$H_1(S^3\backslash K_i; \mathbb{Q}[t, t^{-1}]) \otimes_{\mathbb{Q}[t,t^{-1}]} S^{-1}\mathbb{Z}\Lambda \cong H_1(S^3\backslash K_i; S^{-1}\mathbb{Z}\Lambda).$$

Restricting to $S^{-1}\mathbb{Z}\Lambda = \mathbb{Q}\Lambda$ for simplicity, for any $x, y \in H_1(S^3\backslash K_i; \mathbb{Q}[t, t^{-1}])$,

$$\mathcal{B}l^{M_L}_{\mathbb{Q}\Lambda}(i_*(x \otimes 1), i_*(y \otimes 1)) = \overline{\phi}_i(\mathcal{B}l^i_0(x, y))$$

where $\mathcal{B}l^{M_L}_{\Lambda}$ is the Blanchfield form on M_L induced by ϕ, $\mathcal{B}l^i_0$ is the classical Blanchfield form on the classical rational Alexander module of K_i, and

$$\overline{\phi}_i : \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}] \to \mathcal{K}\Lambda/\mathbb{Q}\Lambda$$

is the monomorphism induced by $\phi : \mathbb{Z} \to \Lambda$ sending 1 to $\phi(\alpha_i)$.

Remarks: Under our hypotheses the coefficient system ϕ extends over the cobordism E, as in the discussion preceding Lemma 2.4, and there is a unique induced coefficient system ϕ_R on M_R. By Property (4) of Lemma 2.6, α_i and its longitudinal push-off α_i^+ are isotopic in E so $\phi((\alpha_i)^+) = \phi_R(\alpha_i)$. Thus $\phi((\alpha_i)^+) \neq 1$ if and only if $\phi_R(\alpha_i \neq 1)$. Moreover, since the meridian of K_i is equated to $(\alpha_i)^+$, $\phi_i(\mu_i) = \phi((\alpha_i)^+) = \phi_R(\alpha_i)$.

The following is perhaps the key result of the paper, that we use to establish certain "injectivity" as discussed in the first paragraph of this section. Recall that the notions of *(n)-solvable* and *rationally (n)-solvable* are defined in Section 6. For the reader who is just concerned with proving that knots and links are not slice, replace the hypothesis below that "W is a rational (k)-solution for M_L" with the hypothesis that "L is a slice link and W is the exterior in B^4 of a set of slice disks for L". Such an exterior is a rational (k)-solution for any k.

Theorem 3.4. *Suppose $L = R(\alpha_i, K_i)$ is obtained by infection. Let the zero surgeries on R, L, and K_i be denoted M_R M_L, M_i respectively. Suppose $\alpha_i \in \pi_1(M_R)^{(k-1)}$ for all i. Suppose W is a rational (k)-solution for M_L, Λ is a PTFA group such that $\Lambda^{(k)} = 1$, and $\psi : \pi_1(W) \to \Lambda$ is a nontrivial coefficient system whose restriction to $\pi_1(M_L)$ is denoted ϕ. Let $A = \{i \mid \phi((\alpha_i)^+) \neq 1\}$. For each $i \in A$, let P_i be the kernel of the composition*

$$\mathcal{A}_0(K_i) \xrightarrow{id\otimes 1} (\mathcal{A}_0(K_i) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Lambda) \xrightarrow{i_*} H_1(M_L; \mathbb{Q}\Lambda) \xrightarrow{j_*} H_1(W; \mathbb{Q}\Lambda).$$

Then $P_i \subset P_i^{\perp}$ with respect to $\mathcal{B}l^i_0$, the classical Blanchfield linking form on the rational Alexander module, $\mathcal{A}_0(K_i)$, of K_i.

Remark: Under the hypotheses of Theorem 3.4, the coefficient system extends over the cobordism E of Figure 2.1 and hence extends to $\pi_1(M_R)$. If this extension is (sloppily) also called ϕ then $\phi(\alpha_i) = \phi((\alpha_i)^+)$ since α_i and its longitude $(\alpha_i)^+$ are isotopic in M_R and hence freely homotopic in E.

Proof of Theorem 3.4. We need:

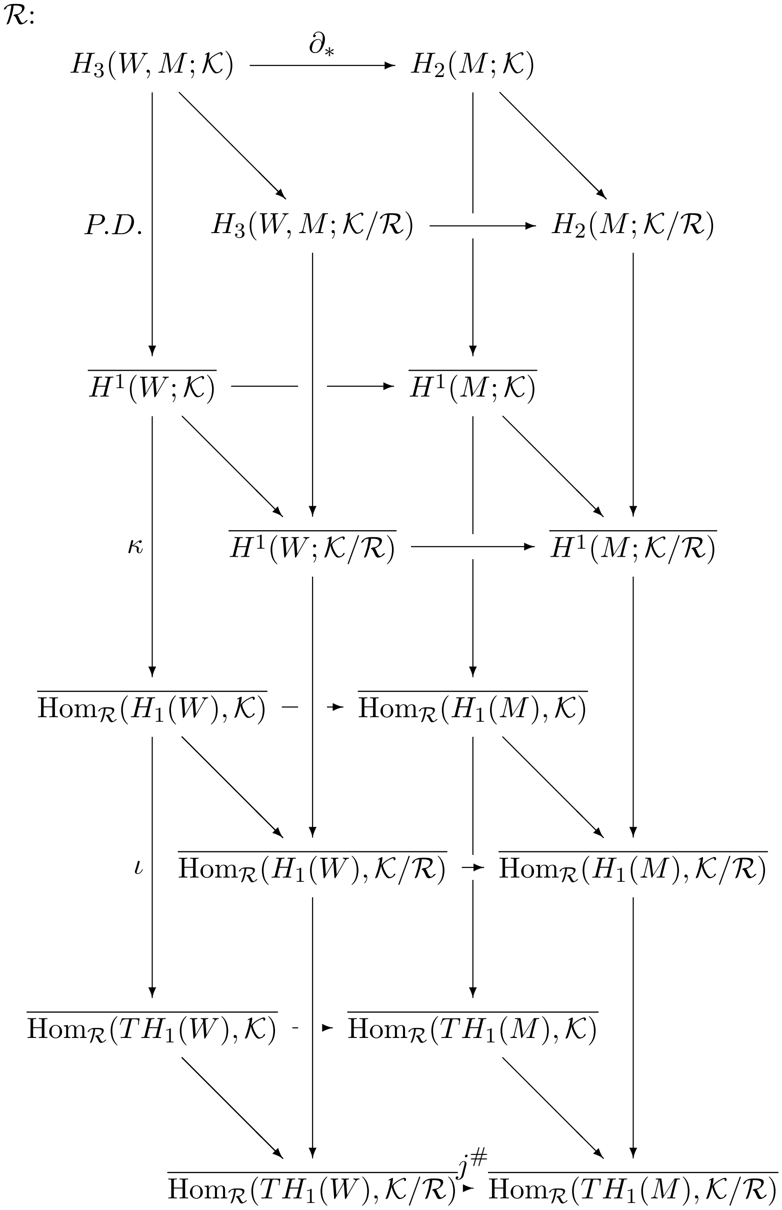
Lemma 3.5. *There is a Blanchfield form, $\mathcal{B}l^{rel}$,*

$$\mathcal{B}l^{rel}:\ TH_2(W,\partial W;\mathcal{R}) \to TH_1(W)^{\#}$$

such that the following diagram, with coefficients in $\mathcal{R}$ unless specified otherwise, is commutative up to sign:

$$\begin{array}{ccc}
TH_2(W,\partial W;\mathcal{R}) & \xrightarrow{\partial_*} & TH_1(M;\mathcal{R}) \\
\Big\downarrow \mathcal{B}l^{rel}_{\mathcal{R}} & & \Big\downarrow \mathcal{B}l^{M}_{\mathcal{R}} \\
TH_1(W;\mathcal{R})^{\#} & \xrightarrow{j^{\#}} & TH_1(M;\mathcal{R})^{\#}
\end{array}$$

Proof of Lemma 3.5. (See also [4, Lemmas 3.2, 3.3]) Consider the following commutative diagram where homology and cohomology is with $\mathcal{R}$ coefficients unless specified and $\mathcal{K}$ denotes the quotient field of $\mathcal{R}$:



where ι is the map induced from the inclusion map of the torsion submodule. Since

$$\overline{\mathrm{Hom}_{\mathcal{R}}(TH_1(W;\mathcal{R}),\mathcal{K})} = 0,$$

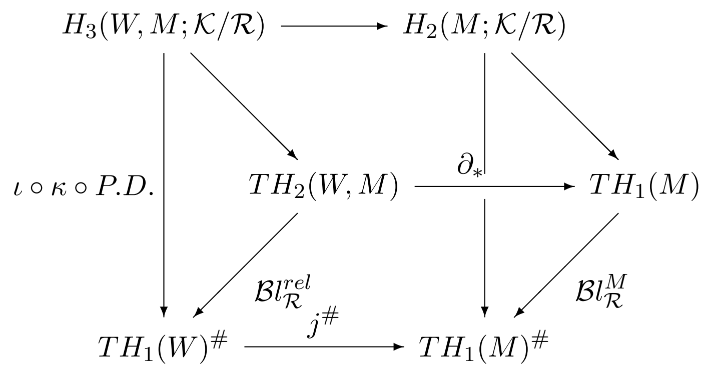
it follows that the image of $H_3(W,M;\mathcal{K}) \to H_3(W,M;\mathcal{K}/\mathcal{R})$ is contained in the kernel of the composition $\iota \circ \kappa \circ P.D.$. Furthermore, from the exact sequence,

$$H_3(W,M;\mathcal{K}) \xrightarrow{\pi} H_3(W,M;\mathcal{K}/\mathcal{R}) \to H_2(W,M;\mathcal{R}) \to H_2(W,M;\mathcal{K})$$

since $H_2(W,M;\mathcal{K})$ is $\mathcal{R}$-torsion-free, $TH_2(W,M;\mathcal{R})$ is isomorphic to the cokernel of π. It follows that there is a well-defined map $\mathcal{B}l^{rel}_{\mathcal{R}} : TH_2(W,M;\mathcal{R}) \to TH_1(W;\mathcal{R})^{\#}$. Similarly, since

$$\overline{\mathrm{Hom}_{\mathcal{R}}(TH_1(M;\mathcal{R}),\mathcal{K})} = 0,$$

there is a well-defined map $\mathcal{B}l^{M}_{\mathcal{R}} : TH_1(M,\mathcal{R}) \to TH_1(M;\mathcal{R})^{\#}$ such that the following diagram commutes.



□

The following result that was proved in [14, Lemma 4.5, Theorem 4.4] in the special case that $\beta_1(M) = 1$. It was proved in more generality in [10, Theorem 6.3] except that there the proof of Lemma 3.5 was omitted.

Lemma 3.6. *Suppose M is connected and is rationally (k)-solvable via W and $\phi : \pi_1(W) \longrightarrow \Lambda$ is a non-trivial coefficient system where Λ is a PTFA group with $\Lambda^{(k)} = 1$. Let $\mathcal{R}$ be an Ore localization of $\mathbb{Z}\Lambda$ so $\mathbb{Z}\Lambda \subset \mathcal{R} \subset \mathcal{K}\Lambda$. Then*

$$TH_2(W,M;\mathcal{R}) \xrightarrow{\partial} TH_1(M;\mathcal{R}) \xrightarrow{j_*} TH_1(W;\mathcal{R})$$

is exact. Moreover, any submodule $P \subset$ kernel j_* satisfies $P \subset (ker\ j_*)^{\perp} \subset P^{\perp}$ with respect to the Blanchfield form on $TH_1(M;\mathcal{R})$.

We can now finish the proof of Theorem 3.4. Suppose $x, y \in P_i$ as in the statement. Let $\mathcal{R} = \mathbb{Q}\Lambda$, $M = M_L$ and let P be the submodule of $H_1(M_L;\mathbb{Q}\Lambda)$ generated by $\{i_*(x\otimes 1), i_*(y\otimes 1)\}$. Then $P \subset$ kernel j_*. Apply Lemma 3.6 to conclude that

$$\mathcal{B}l^{M_L}_{\mathbb{Q}\Lambda}(i_*(x\otimes 1)),(i_*(y\otimes 1)) = 0.$$

By Theorem 3.3,

$$\overline{\phi}_i(\mathcal{B}l^i_o(x,y)) = 0.$$

Since $\overline{\phi}$ is a monomorphism by hypothesis, it follows that $\mathcal{B}l^i_o(x,y) = 0$. Thus $P_i \subset P_i^\perp$ with respect to the classical Blanchfield form on K_i. This concludes the proof of Theorem 3.4. □

4. Iterated Bing doubles and first-order $L^{(2)}$-signatures

In this section we investigate higher-order signature invariants that obstruct any iterated Bing double of K being a topologically slice link. We first state and prove the simplest results and later generalize.

Harvey and Cha-Livingston-Ruberman showed that classical signatures of K, which we call 0^{th}-order signatures, obstruct $BD(K)$ from being a slice link. These signatures vanish if K is an algebraically slice knot. Here we show that certain higher-order signatures of K, similar to Casson-Gordon invariants, that we call **first-order signatures** of K, obstruct $BD(K)$ from being a slice link. To define these, let $G = \pi_1(M_K)$ and $\mathcal{A}_0 = \mathcal{A}_0(K)$ be its classical rational Alexander module. Note that since the longitudes of K lie in $\pi_1(S^3 - K)^{(2)}$,

$$\mathcal{A}_0 \equiv G^{(1)}/G^{(2)} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}[t,t^{-1}]$$

Each submodule $P \subset \mathcal{A}_0$ corresponds to a unique metabelian quotient of G,

$$\phi_P : G \to G/\tilde{P},$$

by setting

$$\tilde{P} \equiv \{x \mid x \in \text{kernel}(G^{(1)} \to G^{(1)}/G^{(2)} \to \mathcal{A}_0/P)\}.$$

Note that $G^{(2)} \subset \tilde{P}$ so $G/\tilde{P}$ is metabelian. Therefore to any such submodule P there corresponds a real number, the Cheeger-Gromov invariant, $\rho(M_K, \phi_P : G \to G/\tilde{P})$.

Definition 4.1. *The first-order $L^{(2)}$-signatures of a knot K are the real numbers $\rho(M_K,\phi_P)$ where $P \subset \mathcal{A}_0(K)$ satisfies $P \subset P^\perp$.*

Since $P = 0$ always satisfies $P \subset P^\perp$, we give a special name to the signature corresponding to this case.

Definition 4.2. *$\rho^1(K)$ of a knot K is the first-order L^2-signature given by the Cheeger-Gromov invariant $\rho(M_K, \phi : G \to G/G^{(2)})$.*

Example 4.3. *Consider the knot K in Figure 4.1. This knot is obtained from the ribbon knot $R = 9_{46}$ by two infections on the band meridians α, β (as in the left-hand side of Figure 1.7). Thus $\{\alpha,\beta\}$ is a basis of $\mathcal{A}_0(K) = \mathcal{A}_0(9_{46})$. There are 3 submodules P for which $P \subset P^\perp$, namely $P_0 = 0$, $P_1 = \langle\alpha\rangle$ and $P_2 = \langle\beta\rangle$. We may apply Lemma 2.4 to show*

$$\rho(M_K,\phi_P) = \rho(M_R,\phi_P) + \epsilon^1_P\rho_0(K_1) + \epsilon^2_P\rho_0(K_2)$$

where ϵ^1 is 0 or 1 according as $\phi_P(\alpha) = 1$ or not (similarly for ϵ^2_P). For our example $\phi_{P_1}(\alpha) = 1$ and $\phi_{P_1}(\beta) \neq 1$. Similarly $\phi_{P_2}(\beta) = 1$ and $\phi_{P_2}(\alpha) \neq 1$. By contrast $\phi_{P_0}(\alpha) \neq 1$ and $\phi_{P_0}(\beta) \neq 1$. Moreover P_1 corresponds to the kernel $\tilde{P}_1$, of $\pi_1(S^3 - R) \to \pi_1(B^4 - \Delta_1)/\pi_1(B^4-\Delta_1)^{(2)}$ for the ribbon disk Δ_1 for R obtained by "cutting the α-band". (Similarly for P_2.) Thus in both cases *the maps ϕ_P on M_R extend over ribbon disk exteriors. Consequently $\rho(M_R, \phi_P) = 0$ for $P = P_1$ and $P = P_\beta$, by Theorem 2.1. Of course $\rho(M_R, \phi_{P_0}) = \rho^1(9_{46})$ by definition. Putting this all together we see that the first-order signatures of the knot K are $\{\rho_0(K_1), \rho_0(K_2), \rho^1(9_{46}) + \rho_0(K_1) + \rho_0(K_2)\}$.*

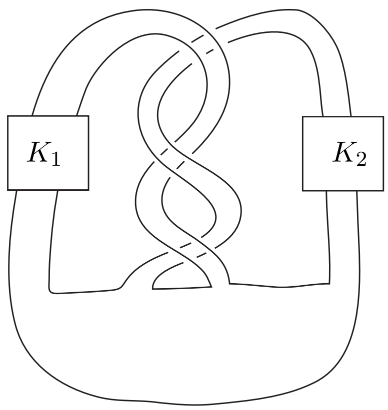
$K =$ 

FIGURE 4.1. A genus 1 algebraically slice knot K

We will now show that the first-order signatures of K, like the ordinary signatures, obstruct any **iterated Bing double** of K from being a (topologically) slice link. This improves on Harvey's theorem which showed this same fact for the integral of the classical signatures [21, Corollary 5.6]. There are several ways to define iterated Bing Doubling. In the most general way, one doubles one component at a time. However for simplicity, let us focus on the notion of Bing doubling wherein we double **every** component. Then the n-fold iterated Bing double of K, $BD^n(K)$, is a 2^n component link. Note that once we show that none of these restricted Bing doubles is slice then it follows that none of the more general iterated Bing doubles is slice.

Theorem 4.4. *Let K be an arbitrary knot. If the n-fold iterated Bing double of K ($n \geq 1$) is topologically slice in a rational homology 4-ball (or more generally is a rationally $(n+1.5)$-solvable link) then one of the first-order signatures of K is zero.*

Corollary 4.5. *If K is the algebraically slice knot of Figure 4.1, with $K_1 = K_2$, and some iterated Bing double of K is a slice link (or even $(n{+}1.5)$-solvable) then $\rho_o(K_1) \in \{0, (-1/2)\rho^1(9_{46})\}$. Therefore if $\rho_0(K_1) \notin \{0, (-1/2)\rho^1(9_{46})\}$ and Arf(K_1)$= 0$, then the n-fold iterated Bing double of K is $(n+1)$-solvable but not slice nor even rationally $(n+1.5)$-solvable. Similarly, if K is the knot of Figure 4.2 where $\rho_0(K') \neq 0$ then no iterated Bing double of K is topologically slice (nor even rationally $(n+1.5)$-solvable).*

Proof of Corollary 4.5. Apply Theorem 4.4 to the knot of Figure 4.1, with $K_1 = K_2$ to conclude that one of the first-order signatures of K must vanish. By Example 4.3, the first-order

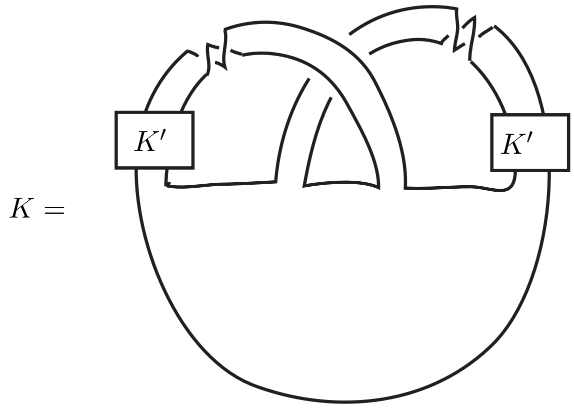


FIGURE 4.2. Order 2 in algebraic concordance group

signatures of K are $\{\rho_0(K_1), 2\rho_0(K_1) + \rho^1(9_{46})\}$. The claimed results follow. Then apply Theorem 4.4 to the knot of Figure 4.2 to conclude that one of the first-order signatures of K must vanish if it is to be slice. But there is only one submodule $P \subset P^\perp$ for the Alexander module of the figure-eight knot, namely $P = 0$. Therefore there is only one first-order signature for the pictured knot K, namely $\rho^1(figure-eight) + 2\rho_0(K')$. Since the figure-eight knot is amphichiral, $\rho^1(figure - eight) = 0$ (this is hown in more detail in [10, Proposition 3.4 and Example 3.5]). □

Proof of Theorem 4.4. Let $L = BD^n(K)$ for some $n \geq 1$ and $M = M_L$. Suppose M is rationally $(n + 1.5)$-solvable via V. We shall show that one of the first-order signatures of K is zero.

Recall that $BD(K)$ can be obtained from the trivial link of two components by infection on the circle α shown dashed in Figure 1.6, using K as the infecting knot. This curve α can be expressed as $[x, y]$ in the fundamental group of the zero surgery on the trivial link where x and y are the meridians. If one now doubles each component of this trivial link, then the image of the curve α becomes a curve that represents the double commutator $[[x, x'], [y, y']]$ for suitably chosen meridians. Continuing in this manner, one sees that the iterated Bing double L can be obtained from the trivial 2^n component link T by a single infection, using the knot K, along a circle α representing, in $\pi_1(M_T)$, an element in $F^{(n)}$ but not in $F^{(n+1)}$. At this point we note that we need not assume that we are dealing with an iterated Bing double, but rather this previous sentence is all that we need assume. Thus our proof is really going to prove:

Theorem 4.6. *Suppose T is a trivial link of m components, $n \geq 1$ and α is an unknotted circle in $S^3 - T$ that represents an element in $F^{(n)} - F^{(n+1)}$ where $F = \pi_1(S^3 - T)$, and L denotes $T(\alpha, K)$, the result of infection of T along α using the knot K. If L is topologically slice in a rational homology 4-ball (or is even a rationally $(n + 1.5)$-solvable link) then one of the first order signatures of K is zero.*

Proceeding with the proof of Theorem 4.6 and hence of Theorem 4.4, since $L = T(\alpha, K)$, there exists a cobordism E as in Figure 2.1 whose boundary is $M_T \sqcup M_K \sqcup -M$. We form a null-bordism W as follows. Cap off $M \subset \partial E$ using V. Thus $\partial W = M_K \cup M_T$. Let $\pi = \pi_1(W)$ and consider $\phi : \pi \to \pi/\pi_r^{(n+2)}$. In the case that V is a slice disk exterior then we can apply Theorem 2.2 to conclude that

$$\rho(M, \phi) = 0.$$

If V is merely a rational $(n+1.5)$-solution, we would like to apply Theorem 6.7 to arrive at the same conclusion. But we must first verify that L satisfies the conditions of Lemma 6.8. This requires only that $\phi(\ell_K) = 1$. This is certainly the case since, by property (5) of Lemma 2.6, ℓ_K is identified with the reverse of meridian of α which bounds a disk in M_T, hence is null-homotopic in W. Let $\overline{\phi}$ be restriction of ϕ to $\pi_1(M_K)$ and ϕ_T denote the restriction of ϕ to $\pi_1(M_T)$. Thus, by Lemma 2.4

$$\rho(M_K, \overline{\phi}) + \rho(M_T, \phi_T) = 0.$$

Since T is a trivial link, $M_T = \partial Y$ where Y is a boundary connected-sum of copies of $S^1 \times B^3$. Since $\pi_1(\partial Y) \cong \pi_1(Y)$, ϕ_T extends to Y. Hence by Theorem 2.2,

$$\rho(M_T, \phi_T) = 0.$$

Therefore

$$\rho(M_K, \overline{\phi}) = 0.$$

It remains only to identify $\rho(M_K, \overline{\phi})$ as one of the first-order signatures of K. First note that the meridian of K is isotopic in E to the infection circle α in M_T. Since $\alpha \in \pi_1(S^3 - T)^{(n)}$, this meridian represents an element of $\pi_1(E)^{(n)}$ and hence an element of $\pi^{(n)}$. Since $G \equiv \pi_1(M_K)$ is normally generated by this meridian,

$$i_*(G) \subset \pi^{(n)}$$

and so

$$i_*(G^{(2)}) \subset \pi^{(n+2)}.$$

Consequently $\overline{\phi}$ factors through $G/G^{(2)}$ and the image of $\overline{\phi}$ is contained in $\pi^{(n)}/\pi_r^{(n+2)}$. By Property 2 of Proposition 2.3, $\rho(M_K, \overline{\phi})$ depends only on the image of $\overline{\phi}$. Thus

$$\rho(M_K, \overline{\phi}) = \rho(M_K, G \to G/G^{(2)} \to G/\tilde{P})$$

where $\tilde{P} = \ker \overline{\phi}$. Therefore we need only characterize $\tilde{P}$. To this end, let $\tilde{\pi} = \pi_1(V)$. From property (1) of Lemma 2.6

$$\pi_1(M) \to \pi_1(E)$$

is surjective with kernel the normal closure of the longitude ℓ_K of K (here we are considering that $S^3 - K \subset M$). Therefore the kernel of the map

$$\tilde{\pi} \to \pi$$

induced by the inclusion $V \hookrightarrow V \cup E$ is the normal closure of ℓ_K. We claim that this induces an isomorphism

$$\tilde{\pi}/\tilde{\pi}_r^{(n+2)} \cong \pi/\pi_r^{(n+2)}.$$

This will follow if we show $\ell_K \in \tilde{\pi}^{(n+2)}$. Recall that $\alpha \in \pi_1(S^3 - T)^{(n)}$. It follows, as shown in [9, Proof of Theorem 8.1] that a stronger fact holds, namely that the longitudinal push-off of α, ℓ_α, lies in $\pi_1(M)^{(n)}$. But ℓ_α is identified to the meridian, μ_K, of $S^3 - K \subset M$. Since $\ell_K \in \pi_1(S^3-K)^{(2)}$ and $\pi_1(S^3-K)$ is normally generated by μ_K,

$$\ell_K \in \pi_1(M)^{(n+2)} \subset \tilde{\pi}^{(n+2)},$$

as required. Hence

$$\tilde{P} = \ker(G \to \tilde{\pi}/\tilde{\pi}_r^{(n+2)}).$$

Moreover, since the copy of S^3-K that is a subset of M_K and the copy of S^3-K that is a subset of M are isotopic in E, we are now free to think of G as π_1 of the latter copy (modulo the longitude).

Now consider $\Lambda = \tilde{\pi}/\tilde{\pi}_r^{(n+1)} \cong \pi/\pi_r^{(n+1)}$ and $\psi : \tilde{\pi} \to \Lambda$. We seek to apply Theorem 3.4 to $L = T(\alpha, K)$, $\alpha \in \pi_1(S^3-T)^{(n)}$, $k = n+1$ and the rational $(n+1)$-solution V for M. To apply Theorem 3.4, we first need to verify that $\psi(\alpha) \neq 1$.

Consider the inclusion $i : M_T \to W$. By property (2) of Lemma 2.6 and since V is a rational (n)-solution, this map induces an isomorphism on $H_1(-;\mathbb{Q})$. By property (3) of Lemma 2.6

$$H_2(W;\mathbb{Q}) \cong H_2(V;\mathbb{Q}) \oplus i_*(H_2(M_K;\mathbb{Q}).$$

Since V is a rational (n)-solution, $H_2(V;\mathbb{Q})$ has a basis consisting of surfaces Σ wherein $\pi_1(\Sigma) \subset \pi^{(n)}$. $H_2(M_K)$ is represented by a capped off Seifert surface $\overline{\Sigma}$ for K. Since $\pi_1(M_K)$ is normally generated by the meridian of K, which lies in $\pi^{(n)}$, $\pi_1(\overline{\Sigma}) \subset \pi^{(n)}$. Thus, by [18, Theorem 2.1], there is a monomorphism

$$i_H : \pi_1(M_T)/\pi_1(M_T)_H^{(n+1)} \hookrightarrow \pi/\pi_H^{(n+1)}$$

where the subscript H denotes Harvey's torsion-free derived series [21, Section 2]. Since the rational derived series is contained in the torsion-free derived series we have the commutative diagram

$$\begin{array}{ccccc} \pi_1(M_T)/\pi_1(M_T)_r^{(n+1)} & \xrightarrow{i_*} & \pi/\pi_r^{(n+1)} & \xrightarrow{\cong} & \Lambda \\ \pi\downarrow & & \downarrow & & \\ \pi_1(M_T)/\pi_1(M_T)_H^{(n+1)} & \xrightarrow{i_H} & \pi/\pi_H^{(n+1)} & & \end{array} \tag{4.1}$$

From this diagram we see that if $\alpha \in \pi_1(M_T)$ mapped to zero in Λ then $\pi(\alpha) = 1$ meaning that $\alpha \in \pi_1(M_T)_H^{(n+1)}$. But this contradicts our hypothesis on α since , for the free group $\pi_1(M_T)$, the torsion-free derived series coincides with the derived series [21, Proposition 2.3]. Hence $\psi(\alpha) \neq 1$ and therefore Theorem 3.4 can be applied. Also note that since $\tilde{\pi}_r^{(n)}/\tilde{\pi}_r^{(n+1)}$ is $\mathbb{Z}$-torsion free, $\psi(\alpha^m) = 1$ only if $m = 0$. We claim that this implies that the kernel of

$$\overline{\phi} : G \to \tilde{\pi}/\tilde{\pi}_r^{(n+2)}$$

is contained in $G^{(1)}$. For suppose that $\overline{\phi}(\mu_K^m c) = 1$ where $c \in G^{(1)}$. Then, since $i_*(G^{(1)}) \subset \pi^{(n+1)}$, $G^{(1)}$ is contained in the kernel of

$$\overline{\psi} : G \xrightarrow{\overline{\phi}} \tilde{\pi}/\tilde{\pi}_r^{(n+2)} \to \tilde{\pi}/\tilde{\pi}_r^{(n+1)},$$

implying that $1 = \overline{\psi}(c\mu_K^m) = \overline{\psi}(\mu_K)^m$. But since $\overline{\psi}(\mu_K) = \psi(\alpha)$, this is a contradiction unless $m = 0$. Thus the kernel of $\overline{\phi}$ is contained in $G^{(1)}$ as claimed.

Now, by Theorem 3.4, if P denotes the kernel of the map

$$\mathcal{A}_0(K) \xrightarrow{i_*} H_1(M;\mathbb{Q}\Lambda) \xrightarrow{j_*} H_1(V;\mathbb{Q}\Lambda).$$

then $P \subset P^\perp$ with respect to the classical Blanchfield form of K. Examine the commutative diagram below where P is the kernel of the bottom horizontal composition. To justify the isomorphism in the bottom row, recall that $H_1(V;\mathbb{Q}\Lambda)$ is identifiable as the ordinary rational homology of the covering space of V whose fundamental group is the kernel of $\psi : \tilde{\pi} \to \Lambda$. Since this kernel is precisely $\tilde{\pi}_r^{(n+1)}$, we have that

$$H_1(V;\mathbb{Q}\Lambda) \cong (\tilde{\pi}_r^{(n+1)}/[\tilde{\pi}_r^{(n+1)},\tilde{\pi}_r^{(n+1)}]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

as indicated in the diagram

$$\begin{array}{ccccccc}
G^{(1)} & \xrightarrow{i_*} & \pi_1(M)^{(n+1)} & \xrightarrow{j_*} & \tilde{\pi}_r^{(n+1)} & \longrightarrow & \tilde{\pi}_r^{(n+1)}/\tilde{\pi}_r^{(n+2)} \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} \\
\mathcal{A}_0(K) & \xrightarrow{i_*} & H_1(M;\mathbb{Q}\Lambda) & \xrightarrow{j_*} & H_1(V;\mathbb{Q}\Lambda) & \xrightarrow{\cong} & (\tilde{\pi}_r^{(n+1)}/[\tilde{\pi}_r^{(n+1)},\tilde{\pi}_r^{(n+1)}]) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}$$

Since, by definition,

$$\tilde{\pi}_r^{(n+2)} \equiv \text{kernel}(\tilde{\pi}_r^{(n+1)} \to (\tilde{\pi}_r^{(n+1)}/[\tilde{\pi}_r^{(n+1)},\tilde{\pi}_r^{(n+1)}]) \otimes_{\mathbb{Z}} \mathbb{Q}))$$

the far-right vertical map j is injective. Thus the kernel of the top horizontal composition is precisely $\pi^{-1}(P)$, which is precisely $\tilde{P}$. This identifies the image of the map $\overline{\phi} : G \to \pi/\pi_r^{(n+2)}$ as $G/\tilde{P}$ for a submodule $P \subset \mathcal{A}_0(K)$ where $P \subset P^\perp$. Thus $\rho(M_K, \overline{\phi})$ is a first-order signature. This completes the proof of Theorem 4.4. □

In examining the proof above, one sees that we made little use of the fact that T was a trivial link. Indeed, with slight modifications, the proof really establishes the following more general result. The more general result says that if one infects a slice link by a knot whose first-order signatures are large then the resulting link is not a slice link. This generalizes Harvey's [21, Theorem 5.4] where it was shown under identical hypotheses that $\rho_0(K)$ obstructs $T(\alpha, K)$ from being slice.

Theorem 4.7. *Suppose T is a slice link of m components, $n \geq 1$ and α is an unknotted circle in $S^3 - T$ with $[\alpha] \in \pi_1(S^3 - T)^{(n)}$ and $[\alpha] \notin \pi_1(M_T)_H^{(n+1)}$. Let L denote $T(\alpha, K)$, the result of infection of T along α using the knot K. If L is topologically slice in a rational homology 4-ball (or is even a rationally $(n+2)$-solvable link) then one of the first order signatures of K is less in absolute value than the Cheeger-Gromov constant of M_T.*

Proof of Theorem 4.7. The following modifications are necessary to the previous proof. We use the fact that V is a (putative) rational $(n+2)$-solution to apply Theorem 6.7 when needed. Then instead of concluding that $\rho(M_K, \overline{\phi}) = 0$ we have only that

$$|\rho(M_K, \overline{\phi})| = |\rho(M_T, \phi_T)| < C_{M_T}.$$

□

Before moving on to more general results, we give another application.

In [21, Section 6] Harvey considered a filtration $\mathcal{F}^m_{(n)}$ of the m-component string link concordance group wherein a string link L is (n)-solvable if its closure $\hat{L}$ is an (n)-solvable link in the sense of [14, Section 8]. The restriction of this filtration to boundary string links, $\mathcal{B}(m)$ was denoted $\mathcal{BF}^m_{(n)}$. Harvey defined specific real-valued higher-order signature invariants, ρ_n of string links. She showed that *each* ρ_n gives a **homomorphism** $\rho_n : \mathcal{B}(m) \to \mathbb{R}$, and induces a homomorphism

$$\rho_n : \mathcal{BF}^m_{(n)}/\mathcal{BF}^m_{(n+1)} \to \mathbb{R}$$

whose image, for any $m > 1$, contains an infinite dimensional vector subspace (over $\mathbb{Q}$) of $\mathbb{R}$. This was slightly improved to $\mathcal{BF}^m_{(n)}/\mathcal{BF}^m_{(n.5)}$ in [18, Theorem 4.5]. From this she concluded that (we incorporate the improvement of [18, Theorem 4.5])

Theorem 4.8. [21, Theorem 6.8] *For any $m > 1$ the abelianization of $\mathcal{BF}^m_{(n)}/\mathcal{BF}^m_{(n.5)}$ has infinite rank, and so $\mathcal{BF}^m_{(n)}/\mathcal{BF}^m_{(n.5)}$ is an infinitely generated subgroup of $\mathcal{F}^m_{(n)}/\mathcal{F}^m_{(n.5)}$*

Our examples cannot be detected by any of Harvey's $\{\rho_n\}$ and so we can show that

Corollary 4.9. *For any $m > 1$, $n \geq 2$, the kernel of Harvey's*

$$\rho_n : \mathcal{BF}^m_{(n)}/\mathcal{BF}^m_{(n.5)} \to \mathbb{R}$$

contains an infinitely generated subgroup.

Proof of Corollary 4.9. Let $\{K_i\}$ be an infinite set of Arf invariant zero knots such that $\{\rho_0(K_i)\}$ is a $\mathbb{Q}$-linearly independent subset of $\mathbb{R}$ (the existence of such a set was established in [15, Proposition 2.6]). Let R_1 be the ribbon knot 9_{46}. It is easy to see, by taking a subset if necessary, that we can assume that $\{\rho_0(K_i), \rho^1(M_{R_1})\}$ is linearly independent. Let J_i denote the knot of Figure 4.1 with $K_1 = K_2 = K_i$. By [15, Proposition 3.1] J_i is a (1)-solvable knot. Fix $m > 1$ and let T denote the trivial m-string link in $D^2 \times I$. Fixing $n \geq 2$, choose a circle $\alpha \in F^{(n-1)} - F^{(n)}$, where F is the group of the exterior of T, such that α bounds a disk in $D^2 \times I$. Let L_i denote $T(\alpha, J_i)$, the string link obtained by infecting T along α using the knot J_i. The closure $\hat{L}_i$ is obtained from the trivial link (which is (n)-solvable) by a (1)-solvable knot along a circle in $F^{(n-1)}$. Thus by Lemma 6.4, $\hat{L}_i$ is (n)-solvable in the sense of [14]. Consequently $L_i \in \mathcal{F}^m_{(n)}$. It is easily seen that L_i is a boundary string link (see [16, Section 2]), so

$$L_i \in \mathcal{BF}^m_{(n)}.$$

It follows directly from Harvey's formula [21, Theorem 5.4] that $\rho_n(L_i) = 0$ (indeed all of her ρ_j vanish for these links). Consider the subgroup of $\mathcal{BF}^m_{(n)}$ generated by $\{L_i\}$. Suppose this were finitely generated. Then there is a subset $\{L_1, ..., L_k\}$ that is a generating set. Consider L_N for some $N > k$. Then the closure of the product $L = L_N L^{\epsilon_1}_{i_1} L^{\epsilon_2}_{i_2} ... L^{\epsilon_q}_q$ is $(n.5)$-solvable for $i_j \in \{1, ..., k\}$ and $\epsilon_j \in \{\pm 1\}$. A crucial point is now the observation that $\hat{L}$ can be obtained from the trivial link by multiple infections on curves α and α_i, all lying in $F^{(n)} - F^{(n+1)}$, where the infection along α_N is done using J_N and the other infections are done using copies of $J_1, ..., J_k$ or their mirror images (if $\epsilon_j = -1$). The proof of Theorem 4.6 applies verbatim to this situation (although it was stated above for only one infection) because the crucial Theorem 3.4 applies to the Alexander module of each infection knot separately. The conclusion is that some first first-order signature of J_N is equal to some linear combination of first-order signatures of the knots $\{J_1, ..., J_k\}$. We saw in Example 4.3 that a first order signature for J_i is an element of the set $\{\rho_0(K_i), \rho^1(R_1) + 2\rho_0(K_i)\}$. It follows that $\rho_0(K_N)$ is a (possibly trivial) linear combination of $\{\rho_0(K_1), ..., \rho_0(K_k), \rho^1(M_{R_1})\}$, contradicting our choice of $\{K_i\}$. Therefore the subgroup of $\mathcal{BF}^m_{(n)}$ generated by $\{L_i\}$ is infinitely generated. □

5. Iterated Bing doubles and higher-order $L^{(2)}$-signatures

The techniques of the proof of Theorem 4.4 and Theorem 4.7 can be generalized to include the iterated Bing doubles of more and more subtle knots, in particular knots whose classical signatures *and* first-order signatures (and Casson-Gordon invariants) vanish. For specificity first consider the family of knots J_n from Figure 1.4. If $n > 1$ these have vanishing classical signatures, first-order signatures and vanishing Casson-Gordon invariants. Yet we show that higher-order signatures obstruct their iterated Bing doubles from being slice. For the family J_n, these higher-order signatures can be calculated, "up to a constant", in terms of the classical signatures of J_0, so we formulate our results terms of $\rho_0(J_0)$ rather than in terms of an n^{th} order signature of J_n. Since the proof will emphasize the structure of J_n as obtained from J_0 by applying an n-fold doubling operator, we will use the notation $J_0 = K$ and $J_n = J_n(K)$.

Theorem 5.1. *Suppose T is a trivial link of m components, k and n are positive integers such that $1 \leq k \leq n$ and α is an unknotted circle in $S^3 - T$ that represents an element in $F^{(k)} - F^{(k+1)}$ where $F = \pi_1(S^3 - T)$, K is a knot with Arf(K) $= 0$, and $L_n(K)$ denotes $T(\alpha, J_{n-k}(K))$, the result of infection of T along α using the knot $J_{n-k}(K)$ shown in Figure 5.1. Then there is a positive constant C such that if $|\rho_0(K)| > C$, then $L_n(K)$ is not topologically slice in a rational homology ball. Moreover, $L_n(K)$ is n-solvable but not rationally $(n + 1)$-solvable. Moreover if $L_n(K)$ is expressed as the closure of the m-component string link $\mathcal{SL}$ then no non-zero multiple of $\mathcal{SL}$ has closure that is rationally $(n + 1)$-solvable.*

Remark 5.2. *The restriction to Arf(K)$= 0$ is only to guarantee that $L_n(K)$ is (n)-solvable. It is not necessary for the conclusion that $L_n(K)$ is* not *$(n + 1)$-solvable. Using the technique of* [10, Theorem 9.1] *one show that $L_n(K)$ is not even rationally $(n.5)$-solvable, and one can choose C independently of n and k (in fact C can be chosen to be the Cheeger-Gromov constant of the zero surgery on the 9_{46} knot).*

Corollary 5.3. *For any K with Arf(K) $= 0$ and any n, there is a constant C such that if if $|\rho_0(K)| > C$ then the Bing double of $J_{n-1}(K)$ is (n)-solvable but not slice nor even rationally $(n + 1)$-solvable.*

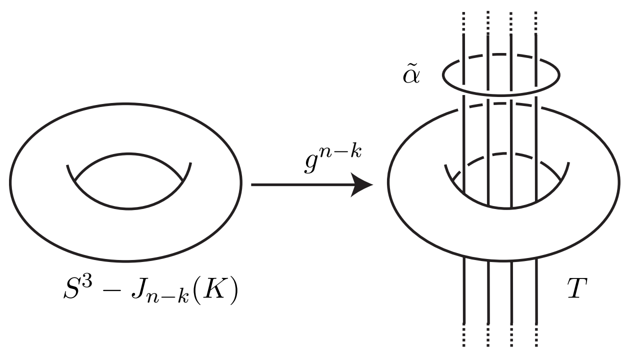


FIGURE 5.1. $L_n(K)$

Corollary 5.4. *Suppose k and n are positive integers, and K is a knot with Arf(K) $= 0$. Then there is a constant C such that if $|\rho_0(K)| > C$, then the k-fold iterated Bing double of $J_{n-k}(K)$) is (n)-solvable but not slice nor even rationally $(n+1)$-solvable.*

Proof of Corollary 5.3. As we have seen in Figure 1.6, a Bing double is obtained by a single infection of the trivial link of two components along a circle α representing the generator of the non-zero group $F^{(1)}/F^{(2)}$ where F is the free group on two letters. The result then follows directly from Theorem 5.1 with $k = 1$. □

Proof of Corollary 5.4. As discussed in the proof of Theorem 4.4, the k-fold iterated Bing double can be obtained from the trivial 2^k component link T by a single infection, using the knot $J_{n-k}(K)$, along a circle α representing, in $\pi_1(M_T)$, an element of$F^{(k)} - F^{(k+1)}$. The result then follows directly from Theorem 5.1. □

Proof of Theorem 5.1. The proof is not substantially different from that of Theorem 4.6, but is notationally much more complicated. Without loss of generality we can assume that $L \equiv L_n(K)$ is the closure of a string link $\mathcal{SL}$ as in the last clause of the theorem. Since the closure of a multiple of $\mathcal{SL}$ is just a particular connected-sum of copies of L, we can (proceeding by contradiction) suppose that $\tilde{L} \equiv \#_{j=1}^{M} L$ were rationally $(n+1)$-solvable for some $M > 0$.

We first establish that L can be obtained from a ribbon link by multiple infections along curves lying in the n^{th}-derived subgroup of the ribbon group. It will follow immediately from Proposition 1.1 that L is (n)-solvable. Specifically let U be the unknot, let $R_i \equiv J_i(U)$ denote the family of ribbon knots obtained recursively by setting $J_0 = U$ in Figure 1.4 or by applying the 9_{46} operator n times to $K = U$ as in Figure 1.7. Then $L_n(U) = T(\alpha, J_{n-k}(U)) = T(\alpha, R_{n-k})$. The proof is postponed, as is the definition of the circles α_*^{n-k}.

Corollary 5.5. *$L_n(K)$ can be obtained from the slice link $L_n(U) = T(\alpha, R_{n-k})$ as the result of 2^{n-k} infections using the knot K each time, along the clones $\alpha_*^{n-k} = \{g^{n-k}(\eta_*^{n-k})\}$ that lie in $\pi_1(S^3 - L_n(U))^{(n)}$. Hence $L_n(K)$ is n-solvable.*

Next we will prove the following very general analog, for links, of [17, Theorem 4.2] (for knots). We can apply Theorem 5.6 to our present situation with $R = T(\alpha, R_{n-k})$, $N = 2^{n-k}$, $K_i = K$ for all i, $L = L_n(K)$. Observe that this will reduce the proof of Theorem 5.1 to proving that the hypotheses of Theorem 5.6 are satisfied for $T(\alpha, R_{n-k})$ and the infection circles α_*^{n-k}. This, in turn, will be accomplished by Theorem 5.7 below. Applying Theorem 5.7 shows that $T(\alpha, R_{n-k})$ satisfies the hypotheses of Theorem 5.6 as desired. Thus the proof of Theorem 5.1 has been reduced to the proofs of the following two theorems and Corollary 5.5. Theorem 5.6 is a very general result while Theorem 5.7 concerns specific links obtained in a recursive fashion.

Theorem 5.6. *Let R be a slice link of m components ($n \geq 1$) and M_R the 0-framed surgery on R. Suppose there exists a collection of homotopy classes*

$$[\eta_i] \in \pi_1(M_R)^{(n)}, \quad 1 \leq i \leq N,$$

that has the following property: For any *rational (n)-solution W of M_R there exists* **some** *i such that $j_*(\eta_i) \notin \pi_1(W)_r^{(n+1)}$ where $j_* : \pi_1(M_R) \to \pi_1(W)$.*

Then, for any oriented trivial link $\{\eta_1, \ldots, \eta_m\}$ in $S^3 \smallsetminus R$ that represents the $[\eta_i]$, and for any N-tuple $\{K_1, \ldots, K_N\}$ of Arf invariant zero knots for which $\rho_0(K_i) > C_{M_R}$ (the Cheeger-Gromov constant of M_R), the link

$$L = R(\eta_1, ..., \eta_N, K_1, ..., K_N)$$

is (n)-solvable but no positive multiple of it is slice (nor even rationally $(n+1)$-solvable). (If the Arf invariant condition is dropped then L is merely rationally n-solvable).

Theorem 5.7. *Let $T_{n-k} \equiv T(\alpha, R_{n-k})$ be as above. Suppose W is an* arbitrary *rational (n)-solution for $M_{T_{n-k}}$. Then at least one of the 2^{n-k} clones $\alpha_*^{n-k} = \{g^{n-k}(\eta_*^{n-k})\}$ maps non-trivially under the inclusion-induced map*

$$j_* : \pi_1(M_{T_{n-k}}) \to \pi_1(W)/\pi_1(W)_r^{(n+1)}.$$

Proof of Theorem 5.6. Supposing that such R and η_i exist, let $L = R(\eta_1, ..., \eta_N, K_1, ..., K_N)$ for knots K_i such that, for each i, $\text{Arf}(K_i) = 0$ and $\rho_0(K_i) > C_{M_R}$ (the Cheeger-Gromov constant of M_R).

Since L is the result of infections on an n-solvable) link along circles lying in the $n^{th}-$ derived subgroup L is n-solvable (merely rationally n-solvable without the Arf invariant condition) by Proposition 1.1.

Now we proceed by contradiction. Suppose that $\tilde{L} \equiv \#_{j=1}^{p} L$ were rationally $(n+1)$-solvable for some $p > 0$. Then there would exist a rational $(n+1)$-solution V with $\partial V = M_{\tilde{L}}$, the zero framed surgery on $\tilde{L}$. Using this we construct a particular rational (n)-solution W for M_R as follows (shown schematically in Figure 5.2). Here C is the standard cobordism from $M_{\tilde{L}}$ to the disjoint union of p copies of M_L. This cobordism is discussed in detail in [15, Section 4]. Cap off the boundary component $M_{\tilde{L}}$ using the rational $(n+1)$-solution V. Since L is obtained from the link R by infection on circles η_i using the knots K_i, there is a cobordism E, as shown in Figure 2.1, such that

$$\partial E = -M_L \sqcup M_R \sqcup_{i=1}^{N} M_i$$

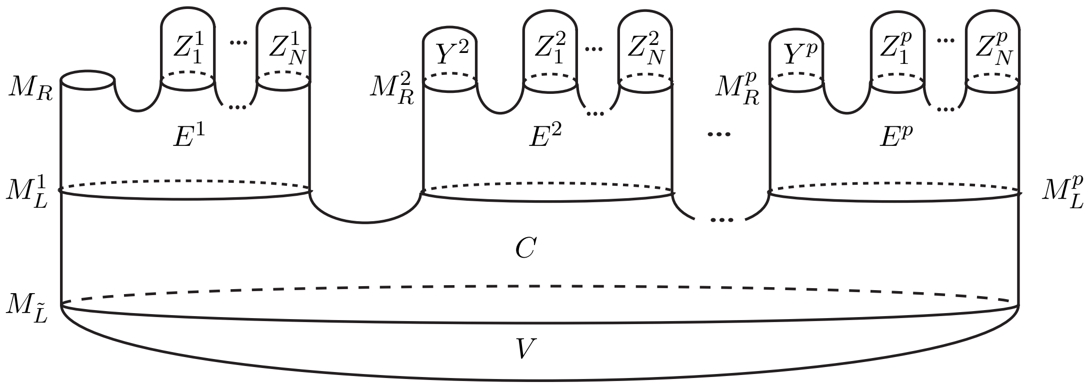


FIGURE 5.2. The rational n-solution W for M_R

where we abbreviate M_{K_i} by M_i. Add a copy of E to each of the p copies of M_L. We denote these copies by $E^j, 1 \leq j \leq p$. Now, for each i, cap off each of the p copies of M_i with a (0)-solution Z_i^j for K_i (we can assume that $\pi_1(Z_i^j) = \mathbb{Z}$ by [15, p.108] [15, Appendix 5]) and cap off each of the copies of M_R, except the "first", with a copy, $Y^j, 2 \leq j \leq p$, of the exterior Y of a set of slicing disks for the slice link R. The resulting manifold W then has a single copy of M_R as its boundary.

Lemma 5.8. *W is a rational n-solution for M_R.*

Proof of Lemma 5.8. By Definition 6.1, we must show that

- $H_1(M_R;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is an isomorphism, and
- W admits a rational (n)-Lagrangian with rational (n)-duals.

First we claim that:

$$H_2(W;\mathbb{Q}) \cong H_2(V;\mathbb{Q}) \oplus_{i,j} H_2(Z_i^j;\mathbb{Q}).$$

Since V is a rational $(n+1)$-solution for $M_{\tilde{L}}$, the inclusion-induced map

$$j_*: \ H_1(M_{\tilde{L}};\mathbb{Q}) \to H_1(V;\mathbb{Q})$$

is an isomorphism. It follows from duality that

$$j_*: \ H_2(M_{\tilde{L}};\mathbb{Q}) \to H_2(V;\mathbb{Q})$$

is the zero map. Therefore if we examine the Mayer-Vietoris sequence with $\mathbb{Q}$-coefficients,

$$H_2(C) \oplus H_2(V) \xrightarrow{\pi_*} H_2(C \cup V) \to H_1(M_{\tilde{L}}) \xrightarrow{(i_*, j_*)} H_1(C) \oplus H_1(V),$$

we see that π_* induces an isomorphism

$$(H_2(C)/(i_*(H_2(M_{\tilde{L}}))) \oplus H_2(V) \cong H_2(V \cup C).$$

Moreover recall that C is obtained from a collar of the disjoint union of p copies of M_L by adding $p-1$ 1-handles (to connect the components) and then adding $m(p-1)$ 2-handles that have the effect of equating pairwise the meridional elements of the copies L. In this way we see that, for any of the boundary components M_L, $H_1(M_L;\mathbb{Q}) \cong H_1(C;\mathbb{Q}) \cong \mathbb{Q}^m$ generated by a set of meridians, and that $H_2(C;\mathbb{Q}) \cong \oplus_{j=1}^{p} H_2(M_L;\mathbb{Q})$ (this is analyzed in more detail in [15, p. 113-114]). It is easy to see that a basis of $i_*(H_2(M_{\tilde{L}}))$ is formed from the sum, $1 \leq j \leq p$ of the elements of natural bases for each $H_2(M_L;\mathbb{Q})$. Thus

$$H_2(V \cup C;\mathbb{Q}) \cong H_2(V;\mathbb{Q}) \oplus (\oplus_{j=1}^{p} H_2(M_L;\mathbb{Q}))/D$$

where $D \cong \mathbb{Q}^m$ is the diagonal subgroup. Now, recall that we have analyzed the homology of E in Lemma 2.6and found that,

$$H_1(M_L) \xrightarrow{i_*} H_1(E)$$

is an isomorphism. Therefore the following Mayer-Vietoris sequence with $\mathbb{Q}$-coefficients is exact,

$$\oplus_{j=1}^{p} H_2(M_L^j) \to \oplus_{j=1}^{p} H_2(E^j) \oplus H_2(V \cup C) \xrightarrow{\pi_*} H_2(V \cup C \sqcup_{j=1}^{p} E^j) \to 0.$$

Moreover, from property (3) of Lemma 2.6,

$$H_2(E) \cong \oplus_{i=1}^{N} H_2(M_i) \oplus H_2(M_R)$$

where the latter $H_2(M_R) \cong H_2(M_L)$ in $H_2(E)$. Combining these facts we have that

$$H_2(V \cup C \sqcup_{j=1}^{p} E^j) \cong H_2(V) \oplus_{j=1}^{p} \oplus_{i=1}^{N} H_2(M_i^j) \oplus_{j=1}^{p} (H_2(M_R^j)/D).$$

The next step in the formation of W was the addition of the slice exteriors Y^j to the copies M_R^j for $2 \leq j \leq p$. Since $H_1(\partial Y^j) \to H_1(Y^j)$ is an isomorphism and $H_2(Y^j) = 0$, the effect on H_2 of adding the Y^j is merely to kill all the H_2 carried by the boundaries $H_2(M_R^j)$, $2 \leq j \leq p$. Taking into account the diagonal relation, we have

$$H_2(V \cup C \cup E^j \cup Y^j) \cong H_2(V) \oplus_{j=1}^{p} \oplus_{i=1}^{N} H_2(M_i^j). \tag{5.1}$$

The final step in the formation of W was the addition of the (0)-solutions Z_i^j to all the copies M_i^j of M_{K_i}. Since, Z_i^j is a (0)-solution, $H_1(M_i^j) \to H_1(Z_i^j)$ is an isomorphism and by duality $H_2(M_i^j) \to H_2(Z_i^j)$ is the zero map. Thus the effect on H_2 of adding the Z_i^j is merely to kill all the generators of the $H_2(M_i^j)$ summand and add $H_2(Z_i^j)$. Thus we have

$$H_2(W;\mathbb{Q}) \cong H_2(V;\mathbb{Q}) \oplus_{i,j} H_2(Z_i^j)$$

This establishes the claim.

Combining some of the observations above it also follows that $H_1(M_R;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is an isomorphism.

We return now to the proof that W is a rational n-solution for M_R. Since V is a rational $(n+1)$-solution, it is a rational (n)-solution. Let $\{\ell_1, \ldots, \ell_g\}$ be a collection of n-surfaces generating a rational n-Lagrangian for V and $\{d_1, \ldots, d_g\}$ be a collection of (n)-surfaces generating its rational (n)-duals. By definition, $2g = \operatorname{rank}_{\mathbb{Q}} H_2(V;\mathbb{Q})$. Similarly, for each i and j take a collection of such (0)-surfaces $\{l_1^{ij}, .., l_k^{ij}\}$, $\{d_1^{ij}, .., d_k^{ij}\}$ for the (0)-solutions Z_i^j. Now taking these surfaces for V together with the collections of surfaces for the Z_i^j, these collections have the required **cardinality** (by the first part of the Lemma) to generate a rational n-Lagrangian with rational (n)-duals for W. Since $\pi_1(V)^{(n)}$ maps into $\pi_1(W)^{(n)}$, the (n)-surfaces for V are also n-surfaces for W. We need to show that the (0)-surfaces for Z_i^j are (n)-surfaces for W.

The group $\pi_1(Z_j^i) \cong \mathbb{Z}$ is generated by the meridian of the knot K_i^j in M_i^j. This meridian is isotopic in E_j to the infection curve $\eta_i^j \in M_R^j$. By hypothesis,

$$[\eta_i^j] \in \pi_1(M_R^j)^{(n)}. \tag{5.2}$$

Therefore

$$j_*(\pi_1(Z_j^i)) \subset \pi_1(W)^{(n)}.$$

Hence *any* surface in Z_i^j is an (n)-surface for W. Moreover, by functoriality of the intersection form with twisted coefficients these collections of surfaces have the required intersection properties to generate a rational n-Lagrangian with rational (n)-duals for W. Hence W is a rational (n)-bordism for M_R, as was claimed.

This completes the proof of Lemma 5.8. □

It also follows from 5.1 above that:

Corollary 5.9. *Let $X = V \cup C \cup E^j \cup Y^j$ so that $\partial X = M_R \cup_{i,j} M_i^j$. Then the cokernel of*

$$H_2(\partial X; \mathbb{Q}) \to H_2(X; \mathbb{Q})$$

is $H_2(V; \mathbb{Q})$.

We continue with the proof of Theorem 5.6. Now set $\Gamma = \pi_1(W)/\pi_1(W)_r^{(n+1)}$. Let $\psi : \pi_1(W) \to \Gamma$ be canonical surjection. Let $\phi : \pi_1(M_R) \to \Gamma$ be the composition $\psi \circ j_*$. Thus by the hypothesis of Theorem 5.6 there exists **some** i such that $\phi(\eta_i) \neq 1$. We shall now compute $|\rho(M_R, \phi)|$ using W, and find it to be greater than C_R. This contradiction will show that in fact $\tilde{L} \equiv \#_{j=1}^p L$ is not rationally $(n+1)$-solvable.

By definition we have,

$$\rho(M_R, \phi) = \sigma_\Gamma^{(2)}(W, \psi) - \sigma(W).$$

By the additivity of the non Neumann and the ordinary signatures ([14, Lemma 5.9]) the latter signatures are the sums of the corresponding signatures for the submanifolds X and Z_i^j.

First consider X. Using Corollary 5.9 and the fact that V is a rational $(n+1)$-solution, X is what is called a rational $(\mathbf{n+1})-$ **bordism** in [10, Section 5]. A rational $(n+1)$-bordism is similar to a rational $(n+1)$-solution except that its boundary need not be connected and the inclusion-induced maps on H_1 from its boundary components are unrestricted. Since $\Gamma^{(n+1)} = 1$, by [10, Theorem 5.9],

$$\sigma_\Gamma^{(2)}(X) - \sigma(X) = 0,$$

as long as each of the boundary components, M, of X satisfies the following alternative: either the induced coefficient system is trivial on $\pi_1(M)$, or

$$\text{rank}_{\mathcal{K}\Gamma} H_1(M; \mathcal{K}\Gamma) = \beta_1(M) - 1. \tag{5.3}$$

This alternative is always satisfied if $\beta_1(M) = 1$ (by [14, Proposition 2.11]), as is the case for each M_i^j. That leaves only M_R to consider. Let $B = \pi_1(X)$. We claim that there is a basis of $H_2(X;\mathbb{Q})$ consisting of surfaces $\Sigma \to X$ for which $\pi_1(\Sigma) \subset B^{(n+1)}$, which is what we call a $B^{(n+1)}$-surface. Recall from 5.1 that $H_2(X;\mathbb{Q})$ is generated by $H_2(V;\mathbb{Q})$ and by the $H_2(M_i^j;\mathbb{Q})$. Since V is a rational $(n+1)$-solution, $H_2(V;\mathbb{Q})$ is generated by $\pi_(V)^{(n+1)}$-surfaces, which are, a fortiori, $B^{(n+1)}$-surfaces. $H_2(M_i^j;\mathbb{Q})$ is generated by a capped-off Seifert surface for the knot K_i^j. Any circle on this Seifert surface lies in $\pi_1(M_i^j)^{(1)}$ and hence lies in $B^{(n+1)}$ since the meridian of M_i^j lies in $B^{(n)}$ as we saw in 5.2. Thus the Seifert surface is also a $B^{(n+1)}$-surface. This completes the verification of the claim. Choose a free group F and a map $F \to \pi_1(M_R)$ inducing an isomorphism on H_1. Now consider the maps

$$F \xrightarrow{i} \pi_1(X) \to B \xrightarrow{\psi} \Gamma.$$

Note that each of these maps induces isomorphisms on $H_1(-;\mathbb{Q})$. Now [18, Proposition 2.11] applies to both $F \to B$ and $\pi_1(M_R) \to B$, since $H_2(X;\mathbb{Q})$ has a basis of $\ker(\psi)$-surfaces since $B^{(n+1)} \subset \ker(\psi)$. Thus

$$H_1(F;\mathcal{K}\Gamma) \cong H_1(M_R;\mathcal{K}\Gamma) \cong H_1(B;\mathcal{K}\Gamma).$$

The rank of the first of these three is known to be $\beta_1(M) - 1$ [14, Lemma 2.12]. This completes the verification that M_R satisfies the alternative 5.3 and hence completes the verification that the the Γ-signature defect of X vanishes.

Now consider the Z_i^j. Let ψ_i^j denote the restriction of ψ to $\pi_1(Z_i^j)$. Then, by definition

$$\sigma_\Gamma^{(2)}(Z_i^j) - \sigma(Z_i^j) = \rho(M_i^j, \psi_i^j).$$

However, since $\pi_1(Z_i^j) \cong \mathbb{Z}$, ψ_i^j factors through $\mathbb{Z}$. Hence by Properties 2,3 and 4 of Proposition 2.3

$$\rho(M_i^j, \psi_i^j) = \rho_0(K_i)$$

if $\psi_i^j(\eta_i^j) \neq 1$ and is zero if $\psi_i^j(\eta_i^j) = 1$. Note that here we have used the fact that the infection circle η_i^j (in M_R^j) is isotopic (in E_j) to the meridian of K_i^j in M_i^j (see property (4) of Lemma 2.6).

Putting all of these together we have

$$\rho(M_R, \phi) = \sum_{i=1}^{N} d_i \rho_0(K_i)$$

where d_i is the number of values of j for which $\psi(\eta_i^j) \neq 1$. Since our hypothesis is that for each i

$$\rho_0(K_i) > C_{M_R},$$

this is a contradiction unless each $d_i = 0$. However recall W is a rational (n)-solution for M_R by Lemma 5.8. Thus by hypothesis there exists **some** i such that $j_*(\eta_i^1) \notin \pi_1(W)_r^{(n+1)}$ where $j_* : \pi_1(M_R) \to \pi_1(W)$. Hence for some i,

$$\psi_i^j(\eta_i^1) \neq 1,$$

so $d_i > 0$. This is a contradiction, completing the proof of Theorem 5.6. $\square$

Thus the proof of Theorem 5.1 has been reduced to the proof of Theorem 5.7 and Corollary 5.5.

To prove Corollary 5.5 we will show that $L_n(K)$ has a variety of different descriptions due to its "fractal" nature. Recall U denotes the trivial knot, and $J_0(K) \equiv K$. First we establish that $J_n(K)$ has an alternative description as the result of 2^n infections on the ribbon knot $R_n = J_n(U)$ using the knot K as the infecting knot each time, along curves that lie in $\pi_1(S^3 \backslash R_n)^{(n)}$. This will be established as part of a much more general result that says that $J_n(K)$ has many alternative descriptions.

To this end note that if K is the trivial knot U then it is easily seen by induction that each $J_n(U)$ is a ribbon knot that we denote R_n, $n \geq 0$, as shown in Figure 5.3 (set $R_0 = U$). For, if R_{n-1} is a ribbon knot then 2 parallels of it form a 2-component ribbon link. Then R_n is formed from this ribbon link by fusing together the 2 components using a knotted band.

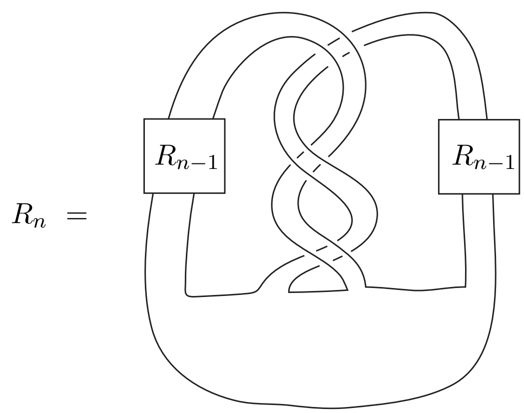


FIGURE 5.3. The recursive family of ribbon knots R_n

Now note that, for each $1 \leq i \leq n$, because of the alternative description of infection as described in Section 1, there are two inclusion maps

$$f^i_{\pm} : S^3 - R_{i-1} \longrightarrow S^3 - R_i$$

as suggested by Figure 5.4.

Let η^0 denote the meridian of R_0, the trivial knot. Let η^1_+, η^1_- denote the two images $f^1_{\pm}(\eta^0)$ in $S^3 - R_1$. We call these **clones** of η^0. More generally, let $\{\eta^i_*\}$ denote the set of 2^i images of η^0 under the 2^i compositions $f^i_{\pm} \circ \cdots \circ f^1_{\pm}$. Note that the induced maps

$$(f^i_{\pm})_* : \pi_1(S^3 \backslash R_{i-1}) \longrightarrow \pi_1(S^3 \backslash R_i)$$

have images contained in the commutator subgroup. Thus the composition

$$(f^i_{\pm})_* \circ \cdots \circ (f^1_{\pm})_* : \pi_1(S^3 \backslash R_0) \longrightarrow \pi_1(S^3 \backslash R_1)^{(1)} \longrightarrow \ldots \longrightarrow \pi_1(S^3 \backslash R_i)^{(i)}$$

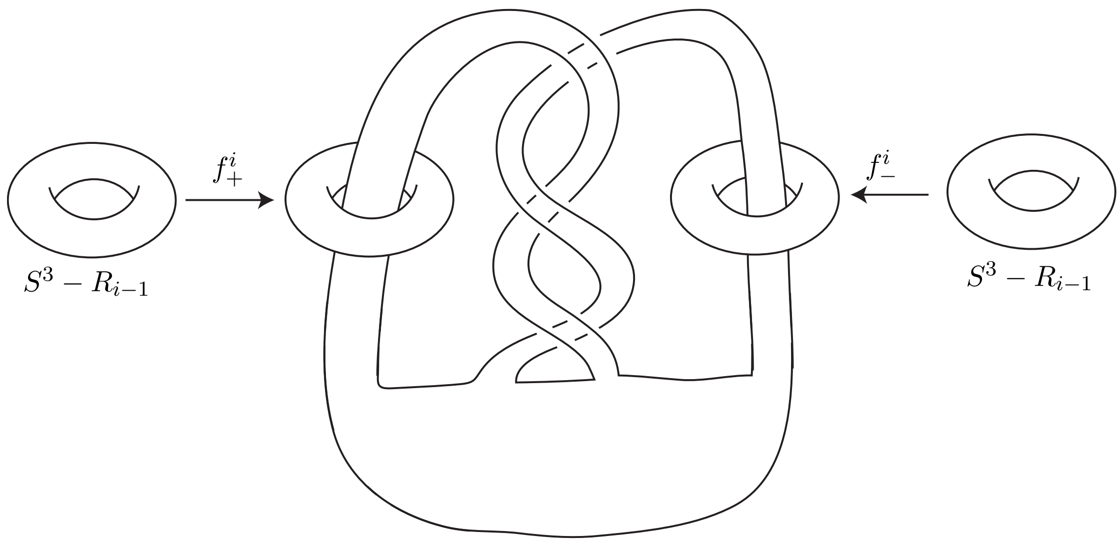


FIGURE 5.4. The embeddings $S^3 - R_{i-1} \hookrightarrow S^3 - R_i$

has image in $\pi_1(S^3\backslash R_i)^{(i)}$. Therefore we see that each of the clones $\{\eta^i_*\}$ lies in $\pi_1(S^3\backslash R_i)^{(i)}$ and in particular each of the clones $\{\eta^n_*\}$ lies in $\pi_1(S^3\backslash R_n)^{(n)}$. The superscript i of $\{\eta^i_*\}$ can serve to remind the reader in which term of the derived series it lies.

The following establishes that $J_n(K)$ has a variety of different descriptions.

Proposition 5.10. *For any knot K and i, $0 \leq i \leq n$, $J_n(K)$ can be obtained from R_i by multiple infections along the 2^i clones*

$$\{\eta^i_*\} = \{f^i_\pm \circ \cdots \circ f^1_\pm(\eta^0)\},$$

using knot $J_{n-i}(K)$ as the infecting knot in each case, and each clone η^i_* lies in $\pi_1(S^3 - R_i)^{(i)}$.

Proof. We proceed by "induction" on i. In the base case, $i = 0$, for any n, there is only one clone, namely η^0 itself. Then the claim is merely that if one infects the unknot by $J_n(K)$ along a meridian then the result is $J_n(K)$, which is obviously true.

Assume that the Proposition is true for some fixed $i-1$ for *any* n such that $n \geq i-1$. Then consider fixed i and arbitrary n subject to $n \geq i$. Recall that $S^3 - J_n(K)$ can be obtained by deleting the two solid tori as shown in the Figure 5.5 and replacing them with two copies of $S^3 - J_{n-1}(K)$. By the inductive hypothesis for (n-1,i-1), $S^3 - J_{n-1}$ can be obtained from $S^3 - R_{i-1}$ by infections on the 2^{i-1} clones $\{\eta^{i-1}_*\} \equiv \{f^{i-1}_\pm \circ \cdots \circ f^1_\pm(\eta^0)\}$ (shown schematically by the very small solid tori in Figure 5.6) using the knot $J_{n-i}(K)$ as the infecting knot in each case. Thus replacing the 2^i solid tori shown in Figure 5.6 by copies of $S^3 - J_{n-i}(K)$ yields $S^3 - J_n$. If we alter our point of view by *postponing* (ignoring for the moment) the infections, then we are precisely in the situation of Figure 5.4, that is if we first replace the two fat solid tori by two copies of $S^3 - R_{i-1}$ (by convention the maps are named $f^i_\pm : S^3 - R_{i-1} \to S^3 - R_i$), then we arrive, by definition, at R_i. The two collections of images in $S^3 - R_i$ of the 2^{i-1} clones

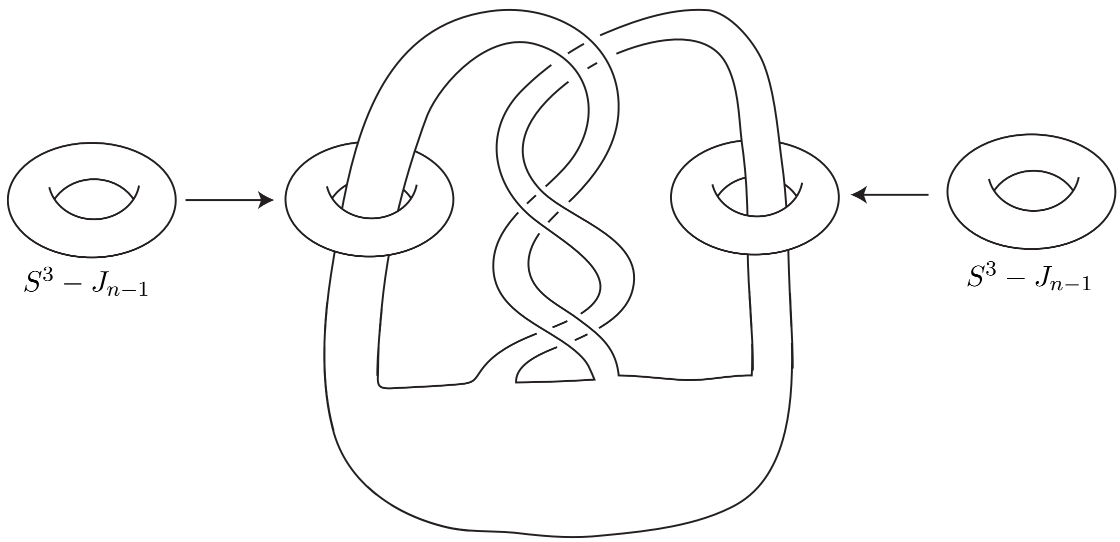


FIGURE 5.5. One definition of $S^3 - J_n$

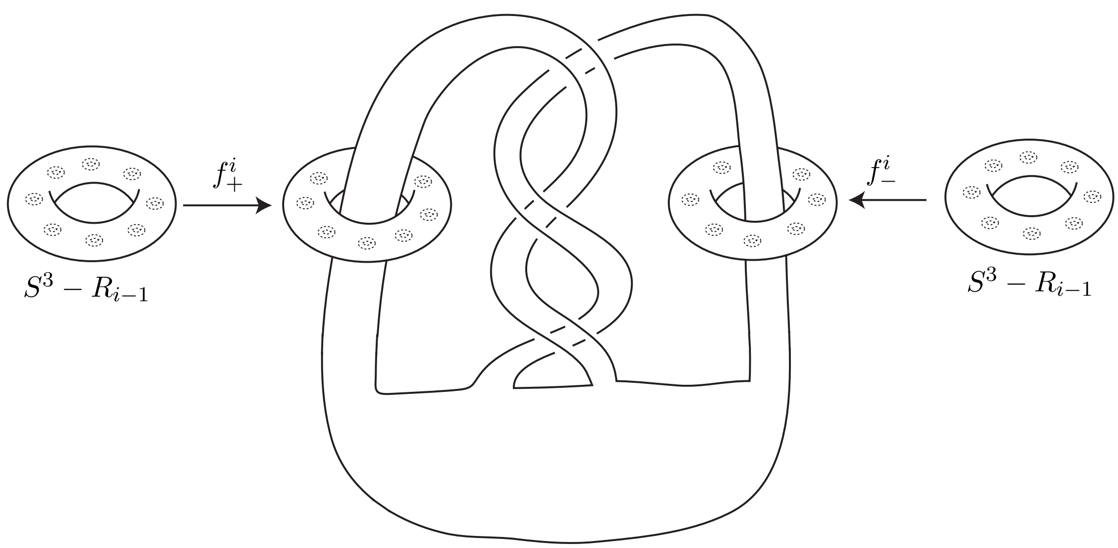


FIGURE 5.6. J_n as the result of 2^i infections on R_i

are precisely the 2^i clones $\{\eta^i_*\} \equiv \{f^i_\pm \circ \cdots \circ f^1_\pm(\eta^0)\}$. If we *then* perform these 2^i infections using the knot $J_{n-i}(K)$ as the infecting knot in each case, we arrive at the description claimed in the Proposition. This completes the inductive step. $\square$

Corollary 5.11. *$J_n(K)$ may be obtained from the ribbon knot R_n as the result of 2^n infections along clones, $\{f^n_\pm \circ \cdots \circ f^1_\pm(\eta^0)\}$, that lie in $\pi_1(S^3 \backslash R_n)^{(n)}$, using the knot K as the infecting knot each time.*

Proof of Corollary 5.11. Apply Proposition 5.10 in the case $i = n$. □

Returning to the proof of Corollary 5.5, suppose that we view the trivial link, T, the positive integer k and the curve $\alpha \in F^{(k)} - F^{(k+1)}$ as fixed. Then $T(\alpha, -)$ may be thought of as an operator from knots to m-component links. From this viewpoint, the proof of Proposition 5.12 below is merely to apply this operator to the result of Proposition 5.10 above. More details are given below.

Proposition 5.12. *For any knot K, and any j, n such that $k \le j \le n$, $L_n(K)$ can be obtained from $L_j(U)$ by multiple infections along the 2^{j-k} clones $\alpha^{j-k}_* = \{g^{j-k}(\eta^{j-k}_*)\}$, using the knot $J_{n-j}(K)$ as the infecting knot in each case, and the clones lie in $\pi_1(S^3 - L_j(U))^{(j)}$.*

Corollary 5.5 follows immediately.

Proof that Proposition 5.12 implies Corollary 5.5. Apply Proposition 5.12 with $j = n$. We claim that $L_n(U)$ is a slice link since it is obtained from the slice link T by infecting using the slice knot R_{n-k} (this is an easy exercise for the reader). □

Proof of Proposition 5.12. By definition,

$$L_n(K) \equiv T(\alpha, J_{n-k}(K)),\ L_j(U) \equiv T(\alpha, J_{j-k}(U)).$$

Since $0 \le j - k \le n - k$, we have from Proposition 5.10 that $J_{n-k}(K)$ can be obtained from $J_{j-k}(U) \cong R_{j-k}$ by multiple infections along the 2^{j-k} clones $\{\eta^{j-k}_*\}$, using the knot $J_{n-j}(K)$ as the infecting knot in each case. Moreover each clone η^{j-k}_* lies in $\pi_1(S^3 - R_{j-k})^{(j-k)}$. Therefore, postponing the infections as in Proposition 5.10, and as suggested by Figure 5.7, we see that $L_n(K) \equiv T(\alpha, J_{n-k}(K))$ can be obtained from $L_j(U) \equiv T(\alpha, R_{j-k})$ by multiple infections along the circles $\{\alpha^{j-k}_*\} = \{g^{j-k}(\eta^{j-k}_*)\}$ (that we shall also call **clones**) using the knot $J_{n-j}(K)$ as the infecting knot in each case.

Since $\alpha \in \pi_1(S^3 - T)^{(k)}$, the technical result [9, proof of Theorem 8.1] shows that the longitudinal push-off, α^+, of α lies in $\pi_1(S^3 - \alpha)^{(j)}$ and thus in $\pi_1(S^3 - T(\alpha, R_{j-k}))^{(k)}$. Hence, since the meridian of R_{j-k} is identified with α^+,

$$g^{j-k}_*(\pi_1(S^3 - R_{j-k})) \subset \pi_1(S^3 - L_j(U))^{(k)},$$

(recalling that $L_j(U) \equiv T(\alpha, J_{n-k}(U)) \equiv T(\alpha, R_{n-k})$). Since, by Proposition 5.10, each clone η^{j-k}_* lies in $\pi_1(S^3 - R_{j-k})^{(j-k)}$, each clone $g^{j-k}(\eta^{j-k}_*)$ lies in $\pi_1(S^3 - L_j(U))^{(j)}$. This completes the proof of Proposition 5.12. □

Finally we are reduced to the proof of Theorem 5.7.

Proof of Theorem 5.7.

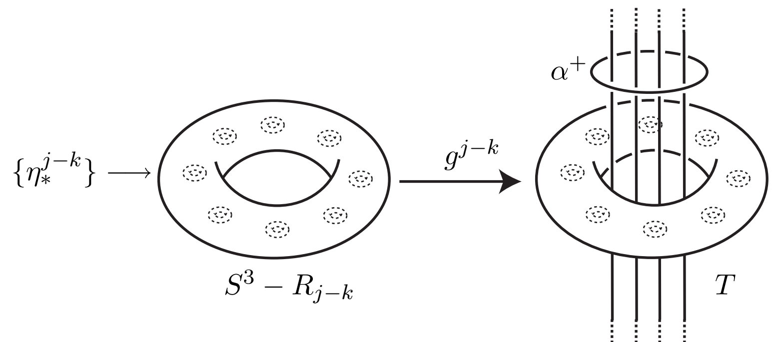


FIGURE 5.7. $T(\alpha, J_{n-k}(K))$ obtained from $T(\alpha, R_{j-k})$

Definition 5.13. *Let μ_j denote a meridian of R_j for $0 \leq j \leq n-k$. A ghost of μ_j , denoted $(\mu_j)_*$ is an element of the set of 2^{n-k-j} circles $\{g^{n-k} f_{\pm}^{n-k} \circ \cdots \circ f_{\pm}^{j+1}(\mu_j)\}$. Thus, for any j, the ghosts of μ_j live in $S^3 - T(\alpha, R_{n-k})$ and represent elements of $\pi_1(S^3 - T(\alpha, R_{n-k}))^{(n-j)}$. These circles are precisely the meridians of the* ***copies*** *of $S^3 - R_j$ that are embedded in $S^3 - T(\alpha, R_{n-k})$ via the maps $\{g^{n-k} f_{\pm}^{n-k} \circ \cdots \circ f_{\pm}^{j+1}\}$. Note that μ_0 is the meridian of $R_0 = U$ so $\mu_0 = \eta^0$. Thus in particular, taking $j = 0$, the ghosts of μ_0 coincide with the clones $\{\alpha_*^{n-k}\}$, that is $\{(\mu_0)_*\} = \{\alpha_*^{n-k}\}$.*

Theorem 5.7 is a special case ($j = 0$) of the following more general result. This Proposition should be viewed as a formulation of the inductive proof of Theorem 5.7.

Proposition 5.14. *Suppose $0 \leq j \leq n-k$ and W is an* arbitrary *rational $(n-j)$-solution for $T_{n-k} \equiv T(\alpha, R_{n-k})$. Then at least one of the ghosts of μ_j maps non-trivially under the inclusion-induced map*

$$j_* : \pi_1(M_{T_{n-k}}) \to \pi_1(W)/\pi_1(W)_r^{(n-j+1)}.$$

Proof of Proposition 5.14. Here we view k and n as fixed and proceed by downward induction on j. First suppose $j = n-k$. In this degenerate case the single ghost is merely the meridian of R_{n-k} viewed as a circle in $T(\alpha, R_{n-k})$, which is of course identified with a push-off, α^+, of α itself, and W is a rational (k)-solution for $M_{T_{n-k}}$. We must show that $j_*(\alpha^+) \neq 1$ under the map

$$j_* : \pi_1(M_{T_{n-k}}) \to \pi_1(W)/\pi_1(W)_r^{(k+1)}.$$

Since T_{n-k} is obtained from the trivial link T by infection on a curve $\alpha \in F^{(k)}$, by [23, Proposition 3.1], there is a degree one map $r : M_{T_{n-k}} \to M_T$ that induces an isomorphism

$$\pi_1(M_{T_{n-k}})/(\pi_1(M_{T_{n-k}}))_r^{(k+1)} \cong F/F^{(k+1)}$$

and sends α^+ to α. Since α is not in $F^{(k+1)}$, $\alpha^+ \neq 1$ in $\pi_1(M_{T_{n-k}})/\pi_1(M_{T_{n-k}})_r^{(k+1)}$. This also implies that the successive terms of the derived series of $\pi_1(M_{T_{n-k}})$ agree with those of the free group (up to this value of k). Thus the derived series, the rational derived series and even Harvey's torsion-free derived series agree for this group (up to this value of k) [21, Section 2]) [21, Proposition 2.3]. This is useful because we now claim that the following is a monomorphism

$$\pi_1(M_{T_{n-k}})/\pi_1(M_{T_{n-k}})_r^{(k+1)} \xrightarrow{j_*} \pi_1(W)/\pi_1(W)_r^{(k+1)}$$

because the composition

$$\pi_1(M_{T_{n-k}})/\pi_1(M_{T_{n-k}})_r^{(k+1)} \xrightarrow{j_*} \pi_1(W)/\pi_1(W)_r^{(k+1)} \to \pi_1(W)/\pi_1(W)_H^{(k+1)}$$

is a monomorphism by the following result of the authors. Here we use that W is a rational (k)-solution for $M_{T_{n-k}}$ and that the torsion-free derived series of a free group is the same its rational derived series.

Proposition 5.15. *[Proposition 4.11 [18]] If M is rationally $(k)-solvable$ via W then, letting $A = \pi_1(M)$ and $B = \pi_1(W)$, the inclusion $j : M \to W$ induces a monomorphism*

$$j_* : \frac{\pi_1(M)}{\pi_1(M)_H^{(k+1)}} \hookrightarrow \frac{\pi_1(W)}{\pi_1(W)_H^{(k+1)}}.$$

It follows that $j_*(\alpha^+) \neq 1$ as required by Proposition 5.14. Thus the Proposition holds for $j = n - k$.

Now suppose that the Proposition is true for $j+1$ where $1 \leq j+1 \leq n-k$. We will establish it for j (downwards induction). So consider a rational $(n-j)$-solution, W, for $M_{T_{n-k}}$. Let $\Lambda = \pi_1(W)/\pi_1(W)_r^{(n-j)}$ and let $\psi : \pi_1(W) \to \Lambda$, and $\phi : \pi_1(M_{T_{n-k}}) \to \Lambda$ be the induced coefficient systems. Note that W is *a fortiori* a rational $(n-j-1)$-solution. Therefore the inductive hypothesis applies to W for the value $j+1$ and allows us to conclude that at least one ghost of μ_{j+1} does not map into $\pi_1(W)_r^{(n-j)}$ under the inclusion, that is, we have $\phi((\mu_{j+1})_*) \neq 1$ for some ghost of μ_{j+1}. We will need this fact below.

We can apply Proposition 5.12 with $K = U$ to deduce that $L_n(U)$ $(\equiv T(\alpha, R_{n-k}) \equiv T_{n-k})$ can be obtained from $L_{n-j-1}(U) \equiv T_{n-j-k-1}$ by infections along the clones $\{\alpha_*^{n-j-k-1}\} = \{g^{n-j-k-1}(\eta_*^{n-j-k-1})\}$ using the knot R_{j+1} as infecting knot in each case. Then, in the notation of Theorem 3.4

$$T_{n-k} = T_{n-j-k-1}(\alpha_i^{n-k-j-1}, R_{j+1}^i, 1 \leq i \leq 2^{n-k-j-1})$$

where $(R_{j+1})^i$ is the i^{th} copy of R_{j+1}. Applying Theorem 3.4 we see that, for any clone such that $\phi((\alpha_i^{n-k-j-1})^+) \neq 1$ the kernel, P_i of the composition

$$\mathcal{A}_0(R_{j+1}) \to (\mathcal{A}_0(R_{j+1}) \otimes \mathbb{Q}\Lambda) \xrightarrow{i_*} H_1(M_{T_{n-k}}; \mathbb{Q}\Lambda) \xrightarrow{j_*} H_1(W; \mathbb{Q}\Lambda),$$

satisfies $P_i \subset P_i^{\perp}$. We claim that there exists at least one such clone. For, by definition of infection, when we infect $T_{n-j-k-1}$ along $\alpha_i^{n-k-j-1}$ the push-off or longitude of such a circle, $(\alpha_i^{n-k-j-1})^+$, is identified to the meridian of the i^{th} copy of the infecting knot $(R_{j+1})^i$. This meridian, when viewed as a circle in T_{n-k}, is not a meridian of the abstract knot R_{j+1}, but rather an embedded copy of that meridian in T_{n-k}. Thus $(\alpha_i^{n-k-j-1})^+$, viewed as a circle in T_{n-k}, is, by definition, one of the one of the **ghosts** of μ_{j+1}! But we established above, by our inductive assumption, that for at least one of these ghosts, $\phi((\mu_{j+1})_*) \neq 1$. Thus we have verified that there is at least one such clone (say the i^{th}) for which the hypotheses of Theorem 3.4 apply. We now restrict attention to such a value of i.

The two circles

$$f_{\pm}^{j+1}(\mu_j) \in \pi_1(S^3 - R_{j+1})^{(1)}$$

as shown in the Figure 5.8, form a generating set for $\mathcal{A}_0(R_{j+1})$ (which is isomorphic to $\mathcal{A}_0(R_1)$ and hence nontrivial). From this we can conclude that at least one of the generators is not in

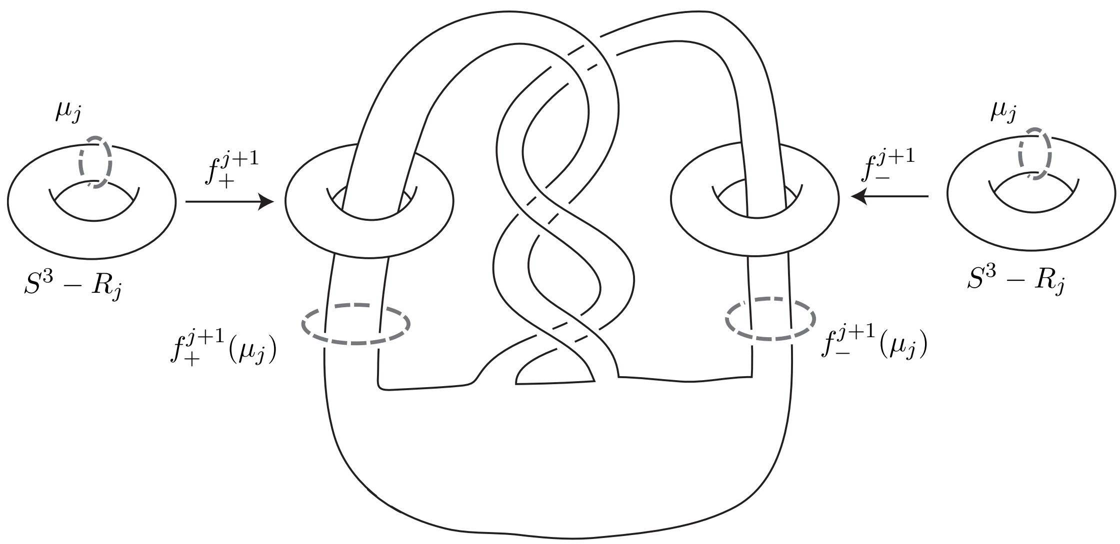


FIGURE 5.8. Inside the i^{th} copy of $S^3 - R_{j+1}$

P_i since otherwise

$$P_i = \mathcal{A}_0(R_{j+1}) \subset \mathcal{A}_0(R_{j+1})^{\perp},$$

contradicting the nonsingularity of the classical Blanchfield form of $\mathcal{A}_0(R_{j+1})$. Finally, consider the commutative diagram below, where we abbreviate $\pi_1(W)$ by π. Recall that $H_1(W;\mathbb{Q}\Lambda)$ is identifiable as the ordinary rational homology of the covering space of W whose fundamental group is the kernel of $\psi : \pi \to \Lambda$. Since this kernel is precisely $\pi_r^{(n-j)}$, we have that

$$H_1(W;\mathbb{Q}\Lambda) \cong (\pi_r^{(n-j)}/[\pi_r^{(n-j)},\pi_r^{(n-j)}]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

as indicated in the diagram below. By the definition of the rational derived series, the far-right vertical map j is injective.

$$\begin{array}{ccccccc}
\pi_1(S^3 - R_{j+1})^{(1)} & \xrightarrow{i_*} & \pi_1(M_{T_{n-k}})^{(n-j)} & \xrightarrow{j_*} & \pi_r^{(n-j)} & \longrightarrow & \pi_r^{(n-j)}/\pi_r^{(n-j+1)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow j \\
\mathcal{A}_0(R_{j+1}) & \xrightarrow{i_*} & H_1(M_{T_{n-k}};\mathbb{Q}\Lambda) & \xrightarrow{j_*} & H_1(W;\mathbb{Q}\Lambda) & \xrightarrow{\cong} & (\pi_r^{(n-j)}/[\pi_r^{(n-j)},\pi_r^{(n-j)}]) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}$$

Hence, since the composition in the bottom row sends one of the two homology classes $[f_\pm^{j+1}(\mu_j)]$ to non-zero, the composition in the top row sends at least one of the two $f_\pm^{j+1}(\mu_j)$ to non-zero under i_*. Now observe that the map i_* in the top row above is induced by one of the compositions $g^{n-k}\circ f_\pm^{n-k}\circ\cdots\circ f_\pm^{j+2}$. Thus

$$i_*(f_\pm^{j+1}(\mu_j)) = g^{n-k}\circ f_\pm^{n-k}\circ\cdots\circ f_\pm^{j+2}\circ f_\pm^{j+1}(\mu_j).$$

For various values of $\pm$ these are precisely the ghosts of μ_j. Hence we have shown that for at least one such ghost of μ_j

$$j_*((\mu_j)_*)\neq 1 \text{ in } \pi_r^{(n-j)}/\pi_r^{(n-j+1)}$$

as desired.

This finishes the inductive proof of Proposition 5.14, hence finishing the proof of Theorem 5.7 and the proof of Theorem 5.1. □

□

□

Since we did not use very heavily the fact that T is a trivial link nor did we use much about the specific nature of the ribbon knot 9_{46}, the proof shows the following more general result.

Theorem 5.16. *Suppose T is a slice link, α is an unknotted circle in S^3-T that represents an element in $\pi_1(S^3-T)^{(k)}$ but not in $\pi_1(M_T)_H^{(k+1)}$. Suppose for each j, $1\leq j\leq n-k$, R_j is a slice knot, $\{\eta_{j1},\ldots,\eta_{jm_j}\}$ is a trivial link of circles in S^3-R_j with the property that the submodule of the classical Alexander polynomial of R_j generated by $\{\eta_{j1},\ldots,\eta_{jm_j}\}$ contains elements x,y such that $\mathcal{B}\ell_0^j(x,y)\neq 0$, where $\mathcal{B}\ell_0^j$ is the Blanchfield form of R_j. Finally suppose that Arf(K)= 0 Then the result, $L(K)\equiv T_\alpha\circ R_{n-k}\circ\cdots\circ R_1(K)$, of the iterated generalized doubling (applied to K) lies in $\mathcal{F}_n$ and there is a constant C, such that if $|\rho_0(K)|>C$, then $L(K)$ is of infinite order in the topological concordance group (moreover no multiple lies in $\mathcal{F}_{n+1}$).*

6. Higher-Order Signatures as Obstructions to being Slice and the COT n-solvable Filtration

The COT n-solvable filtration

Recall that [14, Section 8] introduced a filtration of the concordance classes of links $\mathcal{C}$

$$\cdots\subseteq\mathcal{F}_n\subseteq\cdots\subseteq\mathcal{F}_1\subseteq\mathcal{F}_{0.5}\subseteq\mathcal{F}_0\subseteq\mathcal{C}.$$

where the elements of $\mathcal{F}_n$ and $\mathcal{F}_{n.5}$ are called (n)**-solvable links** and $(n.5)$**-solvable links** respectively. In the case of knots this is a filtration by *subgroups* of the knot concordance group. A slice link L has the property that it's zero surgery M_L bounds a 4-manifold W (namely the exterior of the slicing disks) such that $H_1(M_L)\to H_1(W)$ is an isomorphism and $H_2(W)=0$. An **n-solvable** link is one, loosely speaking, such that M_L bounds a 4-manifold W such that $H_1(M_L)\to H_1(W)$ is an isomorphism and the intersection form on $H_2(W)$ "looks" hyperbolic modulo the n^{th}-term of the derived series of $\pi_1(W)$. We shall only give a detailed definition of the slightly larger class of **rationally** (n)**-solvable links**.

For a compact oriented topological 4-manifold W, let $W^{(n)}$ denote the covering space of W corresponding to the n-th derived subgroup of $\pi_1(W)$. The deck translation group of this cover is the solvable group $\pi_1(W)/\pi_1(W)^{(n)}$. Then $H_2(W^{(n)};\mathbb{Q})$ can be endowed with the structure of a right $\mathbb{Q}[\pi_1 W)^{(n)}]$-module. This agrees with the homology group with twisted coefficients $H_2(W;\mathbb{Q}[\pi_1(W)^{(n)}])$. There is an equivariant intersection form

$$\lambda_n : H_2(W^{(n)};\mathbb{Q}) \times H_2(W^{(n)};\mathbb{Q}) \longrightarrow \mathbb{Q}[\pi_1(W)/\pi_1(W)^{(n)}]$$

[33, Chapter 5][14, Section 7]. The usual intersection form is the case $n = 0$. In general, these intersection forms are singular. Let $I_n \equiv \text{image}(j* : H_2(\partial W^{(n)};\mathbb{Q}) \to H_2(W^{(n)};Q))$. Then this intersection form factors through

$$\overline{\lambda_n} : H_2(W^{(n)};\mathbb{Q})/I_n \times H_2(W^{(n)};\mathbb{Q})/I_n \longrightarrow \mathbb{Q}[\pi_1(W)/\pi_1(W)^{(n)}].$$

We define a *rational n-Lagrangian* of W to be a submodule of $H_2(W;\mathbb{Q}[\pi_1 W)^{(n)}]$ on which $\overline{\lambda}_n$ vanishes identically and which maps onto a $\frac{1}{2}$-rank subspace of $H_2(W;\mathbb{Q})/I_0$ under the covering map. An *n-surface* is a based and immersed surface in W that can be lifted to $W^{(n)}$. Observe that any class in $H_2(W^{(n)})$ can be represented by an n-surface and that λ_n can be calculated by counting intersection points in W among representative n-surfaces weighted appropriately by signs and by elements of $\pi_1(W)/\pi_1(W)^{(n)}$. We say a rational n-Lagrangian L admits *rational m-duals* (for $m \leq n$) if L is generated by (lifts of) n-surfaces $\ell_1, \ell_2, \ldots, \ell_g$ and there exist m-surfaces $d_1, d_2, \ldots, d_g$ such that $H_2(W;\mathbb{Q})/I_0$ has rank $2g$ and $\lambda_m(\ell_i, d_j) = \delta_{i,j}$.

Under the assumption that we will impose below, that

$$H_1(M;\mathbb{Q}) \to H_1(W;\mathbb{Q})$$

is an isomorphism, it follows that the dual map

$$H_3(W, M;\mathbb{Q}) \to H_2(M;\mathbb{Q})$$

is an isomorphism and hence that $I_0 = 0$. Thus the "size" of rational (n)-solutions is dictated by the rank of $H_2(W;\mathbb{Q})$.

Definition 6.1. *Let n be a nonnegative integer. A compact, connected oriented topological 4-manifold W with $\partial W = M$ is a **rational n-solution for** M if*

- *$H_1(M;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is an isomorphism, and*
- *W admits a rational (n)-Lagrangian with rational (n)-duals.*

Then we say that M is rationally (n)-solvable via W. *A link L is an* (n)-solvable link *if M_L is rationally (n)-solvable for some such W.*

Definition 6.2. *Let n be a nonnegative integer. A compact, connected oriented 4-manifold W with $\partial W = M$ is a **rational $n.5$-solution for** M if*

- *$H_1(M;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is an isomorphism, and*
- *W admits a rational n-Lagrangian with rational $(n+1)$-duals.*

*Then we say that M **is rationally $(n.5)$-solvable via** W. A link L is an $(n.5)$**-solvable link** if M_L is rationally $(n.5)$-solvable for some such W.*

A 4-manifold W is an (n)**-solution** (respectively an $(n.5)$**-solution**) if, in addition, it is spin, it satisfies the conditions above with $\mathbb{Q}$ replaced by $\mathbb{Z}$ and the equivariant **self-intersection form** also vanishes on the Lagrangian (see [14, Section 8].

Remark 6.3. *(1) An (n)-solution is a fortiori a rational (n)-solution.*

(2) An (n)-solution (respectively rational (n)-solution) is a fortiori an (m)-solution (respectively rational (m)-solution) for any $m < n$.

(3) If L is slice in a topological (rational) homology 4-ball then the complement of a set of slice disks is an (rational) (n)-solution for any integer or half-integer n. This follows since if $H_2(W;\mathbb{Z}) = 0$ then the Lagrangian may be taken to be the zero submodule.

The following result is useful.

Lemma 6.4. *Suppose L is a link obtained from a $(p+q)$-solvable link R by infection along curves in $\pi_1(S^3 - R)^{(p)}$ using knots K_i. Suppose the knots K_i are (q)-solvable. Then L is also a $(p+q)$-solvable link.*

Proof. One can repeat almost verbatim the proof of [15, Proposition 3.1] (see also [17, Corollary 3.14]). However, one also needs the following result.

Lemma 6.5. *Suppose $\phi : A \to B$ is a group homomorphism that is surjective on abelianizations. Then, for any positive integer n, $\phi(A)$ normally generates $B/B^{(n)}$.*

Proof of Lemma 6.5. The proof is by induction on n. The case $n = 1$ is the hypothesis. Now consider $b \in B$. Then $b = \phi(a)\prod_{i=1}^{m}[b_{i1}, b_{i2}]$ where $a \in A$ and $b_{i1}, b_{i2} \in B$. It now suffices to show that a single commutator

$$[b_1, b_2] \in <\phi(A)> B^{(n)}$$

where $<\phi(A)>$ denotes the normal closure in B. By the inductive hypothesis

$$b_j \in <\phi(A)> B^{(n-1)}$$

for $j = 1, 2$. Hence

$$[b_1, b_2] \in [<\phi(A)> B^{(n-1)}, <\phi(A)> B^{(n-1)}],$$

which equals $<\phi(A)> B^{(n)}$ by simple commutator calculus. □

□

Theorem 6.6. *(Cochran-Orr-Teichner* [14, Theorem 4.2]*) If a knot K is rationally $(n.5)$-solvable via W and $\phi : \pi_1(M_K) \to \Gamma$ is a PTFA coefficient system that extends to $\pi_1(W)$ and such that $\Gamma^{(n+1)} = 1$, then $\rho(M_K, \phi) = 0$.*

For links the following recent result of the first two authors is the best known result (although see [10, Theorem 5.9]). Note the extra rank condition.

Theorem 6.7. *[Cochran-Harvey* [18, Theorem 4.9, Proposition 4.11]*] Let Γ be a PTFA group such that $\Gamma^{(n+1)} = 0$. Let M be a closed, connected, oriented 3-manifold equipped with a non-trivial coefficient system $\phi : \pi_1(M) \to \Gamma$. Suppose $\text{rank}_{\mathcal{K}\Gamma}(H_1(M;\mathcal{K}\Gamma)) = \beta_1(M) - 1$. Then if M is rationally $(n.5)$-solvable via a 4-manifold W over which ϕ extends, then*

$$\rho(M, \phi) = \sigma^{(2)}_{\Gamma}(W) - \sigma(W) = 0.$$

Moreover, if additionally M is rationally $(n+1)$-solvable via W then the extra rank condition above is automatically satisfied.

Proof that Theorem 6.7 implies Theorem 2.2. Since Γ is PTFA, it is solvable so there exists some n such that $\Gamma^{(n+1)} = 0$. Let W denote the exterior of the slicing disks. By Alexander duality, $H_2(W;\mathbb{Q}) = 0$ and $H_1(M_L;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is an isomorphism. Thus W is a certainly a rational $(n+1)$-solution for L. The the result follows immediately from Theorem 6.7. □

There is another common situation in which the extra rank condition is satisfied.

Lemma 6.8. *Suppose L is a link obtained from the link R by infections on circles η_i using knots K_i. Suppose $\phi : \pi_1(M_L) \to \Gamma$ is a nontrivial PTFA coefficient system such that $\phi(\mu_{\eta_i} \equiv l_{K_i}) = 1$. Then there is a coefficient system $\phi : \pi_1(M_L) \to \Gamma$ induced on M_R and*

$$\mathit{rank}_{\mathcal{K}\Gamma}(H_1(M_L;\mathcal{K}\Gamma)) \geq \mathit{rank}_{\mathcal{K}\Gamma}(H_1(M_R;\mathcal{K}\Gamma)).$$

In particular if R is the trivial link of m components then

$$\mathit{rank}_{\mathcal{K}\Gamma}(H_1(M_L;\mathcal{K}\Gamma)) = \beta_1(M_L) - 1.$$

Proof of Lemma 6.8. Consider the cobordism E_L of Figure 2.1. By Property (1) of Lemma 2.6, the map

$$\pi_1(M_L) \to \pi_1(E_L)$$

is a surjection whose kernel is normally generated by $\{\mu_{\eta_i}\}$. Thus, as shown there, ϕ extends uniquely to $\pi_1(E_L)$ and hence by restriction to $\pi_1(M_R)$. Therefore there is a surjection

$$H_1(M_L;\mathcal{K}\Gamma) \to H_1(E_L;\mathcal{K}\Gamma)$$

so

$$\mathrm{rank}_{\mathcal{K}\Gamma}(H_1(M_L;\mathcal{K}\Gamma)) \geq \mathrm{rank}_{\mathcal{K}\Gamma}(H_1(E_L;\mathcal{K}\Gamma)).$$

Now examine the Mayer-Vietoris sequence with $\mathcal{K}\Gamma$ coefficients for E_L as in the proof of Lemma 2.5

$$\oplus_i H_1(\eta_i \times D^2) \to \oplus_i H_1(M_{K_i}) \oplus H_1(M_R) \to H_1(E_L) \xrightarrow{\partial_*} \oplus_i H_0(\eta_i \times D^2).$$

We claim that the inclusion-induced maps

$$H_0(\eta_i \times D^2;\mathcal{K}\Gamma) \longrightarrow H_0(M_i;\mathcal{K}\Gamma)$$

are injective. In the case that $\phi(\eta_i) \neq 1$, $H_0(\eta_i \times D^2;\mathcal{K}\Gamma) = 0$ by [14, Proposition 2.9], so injectivity holds. If $\phi(\eta_i) = 1$ then, since η_i is equated to the meridian of K_i, $\phi(\mu_{K_i}) = 1$. Since μ_i normally generates $\pi_1(M_i)$, it follows that the coefficient systems on $\eta_i \times D^2$ and M_i are trivial and hence the injectivity follows from the injectivity with $\mathbb{Z}$-coefficients, which is obvious since both are path-connected. Hence ∂_* is the zero map. Similarly we claim that the inclusion-induced maps

$$H_1(\eta_i \times D^2;\mathcal{K}\Gamma) \longrightarrow H_1(M_{K_i};\mathcal{K}\Gamma)$$

are isomorphisms. In the case that $\phi(\eta_i) \neq 1$, both groups are zero by [14, Lemma 2.10]. If $\phi(\eta_i) = 1$ then both coefficient systems are trivial and result follows from the result for $\mathbb{Z}$-coefficients, which is obvious since u_{K_i} generates $H_1(M_{K_i}) \cong \mathbb{Z}$.

Armed with these observations, it now follows from the Mayer-Vietoris sequence that

$$H_1(M_R;\mathcal{K}\Gamma) \cong H_1(E_L;\mathcal{K}\Gamma).$$

and the first result follows.

If R is a trivial link then $\pi_1(M_R)$ is the free group F of rank m. But it is easy to see from an Euler characteristic argument ([14, Lemma 2.12]) that

$$\text{rank}_{\mathcal{K}\Gamma}(F;\mathcal{K}\Gamma)) = \beta_1(F) - 1 = m - 1.$$

Thus

$$\text{rank}_{\mathcal{K}\Gamma}(H_1(M_L;\mathcal{K}\Gamma)) \geq \beta_1(M_L) - 1$$

but by [14, Proposition 2.11], this is also the maximum this rank can achieve, so the inequality is an equality. □

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