2-TORSION IN THE n-SOLVABLE FILTRATION OF THE KNOT CONCORDANCE GROUP

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Abstract. Cochran-Orr-Teichner introduced in [11] a natural filtration of the smooth knot concordance group $\mathcal{C}$

$$\cdots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C},$$

called the (n)-solvable filtration. We show that each associated graded abelian group $\{\mathbb{G}_n = \mathcal{F}_n/\mathcal{F}_{n.5} \mid n \in \mathbb{N}\}$, $n \geq 2$ contains infinite linearly independent sets of elements of order 2 (this was known previously for $n = 0, 1$). Each of the representative knots is negative amphichiral, with vanishing s-invariant, τ-invariant, δ-invariants and Casson-Gordon invariants. Moreover each is slice in a rational homology 4-ball. In fact we show that there are many distinct such classes in $\mathbb{G}_n$, one for each "distinct" n-tuple $\mathcal{P} = (p_1(t), ..., p_n(t))$ of knot polynomials. Such a sequence of polynomials records the orders of certain submodules of a sequence of higher-order Alexander modules of the knot.

1. Introduction

A (classical) knot K is the image of a smooth embedding of an oriented circle in S^3. Two knots, $K_0 \hookrightarrow S^3 \times \{0\}$ and $K_1 \hookrightarrow S^3 \times \{1\}$, are concordant if there exists a proper smooth embedding of an annulus into $S^3 \times [0,1]$ that restricts to the knots on $S^3 \times \{0,1\}$. Let $\mathcal{C}$ denote the set of (smooth) concordance classes of knots. The equivalence relation of concordance first arose in the early 1960's in work of Fox, Kervaire and Milnor in their study of isolated singularities of 2-spheres in 4-manifolds and, indeed, certain concordance problems are known to be *equivalent* to whether higher-dimensional surgery techniques "work" for topological 4-manifolds [15][28][3]. It is well-known that $\mathcal{C}$ can be endowed with the structure of an abelian group (under the operation of connected-sum), called the smooth knot concordance group. The identity element is the class of the trivial knot. Any knot in this class is concordant to a trivial knot and is called a slice knot. Equivalently, a slice knot is one that is the boundary of a smooth embedding of a 2-disk in B^4. In general, the abelian group structure of $\mathcal{C}$ is still poorly understood. But much work has been done on the subject of knot concordance (for excellent surveys see [19] and [35]). In particular, [11] introduced a natural filtration of $\mathcal{C}$ by subgroups

$$\cdots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}.$$

2000 *Mathematics Subject Classification.* Primary 57M25; Secondary 20J.

[†]Partially supported by NSF DMS-0706929.

[††]Partially supported by NSF CAREER DMS-0748458.

[†††]Partially supported by NSF DMS-0805867.

called the *(n)-solvable filtration* of $\mathcal{C}$ and denoted $\{\mathcal{F}_n\}$ (defined in Section 3). The non-triviality of $\mathcal{C}$ can be measured in terms of the associated graded abelian groups $\{\mathbb{G}_n = \mathcal{F}_n/\mathcal{F}_{n.5} \mid n \in \mathbb{N}\}$ (here we ignore the other "half" of the filtration, $\mathcal{F}_{n.5}/\mathcal{F}_{n+1}$, where almost nothing is known). This paper is concerned with elements of order two in $\mathcal{C}$ and, more generally, with elements of order two in $\mathbb{G}_n$.

We will review some of the history of 2-torsion phenomena in $\mathcal{C}$ in the context of the n-solvable filtration. One of the earliest results concerning $\mathcal{C}$ was an epimorphism constructed by Fox and Milnor [15]

$$FM : \mathcal{C} \twoheadrightarrow \mathbb{Z}_2^\infty.$$

Soon thereafter, Levine constructed an epimorphism

$$\mathcal{C} \twoheadrightarrow \mathcal{AC} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty, \tag{1.1}$$

to a group, $\mathcal{AC}$, that became known as the *algebraic knot concordance group.* Any knot in the kernel of (1.1) is called an **algebraically slice** knot. In terms of the n-solvable filtration, Levine's result is [11, Remark 1.3.2, Thm. 1.1]:

$$\mathbb{G}_0 \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty.$$

It is known that there exist elements of order two in $\mathcal{C}$ that realize the above 2-torsion invariants, that is to say, the map FM splits. Let $\overline{K}$ denote the **mirror image** of the oriented knot K, obtained as the image of K under an orientation reversing homeomorphism of S^3; and let $r(K)$ denote the **reverse** of K, which is obtained by merely changing the orientation of the circle. Then it is known that $K\#r(\overline{K})$ is a slice knot, so the inverse of $[K]$ in $\mathcal{C}$, denoted $-[K]$, is represented by $r(\overline{K})$, denoted $-K$. A knot K is called **negative amphichiral** if K is isotopic to $r(\overline{K})$. It follows that, for any negative amphichiral knot K, $K\#K$ is a slice knot, since it is isotopic to $K\#-K$. Hence negative amphichiral knots represent elements of order either 1 or 2 in $\mathcal{C}$. It is a conjecture of Gordon that every class of order two in $\mathcal{C}$ can be represented by a negative amphichiral knot [19].

In fact the work of Milnor and Levine in the 1960's resulted in a more precise statement:

$$\mathbb{G}_0 \cong \bigoplus_{p(t)} \left(\mathbb{Z}^{r_p} \oplus \mathbb{Z}_2^{m_p} \oplus \mathbb{Z}_4^{n_p}\right)$$

where the sum is over all primes $p(t) \in \mathbb{Z}[t]$ where $p(t) \doteq p(t^{-1})$ and $p(1) = \pm 1$ [34, Sections 10,11,24][24, p.131]. That is, the algebraic concordance group (and $\mathbb{G}_0$) admits a certain $p(t)$-*primary decomposition*, wherein a knot has a nontrivial $p(t)$-primary part only if $p(t)$ is a factor of its Alexander polynomial.

In the 1970's the introduction of Casson-Gordon invariants in [1][2] led to the discovery that the subgroup of algebraically slice knots was of infinite rank and contained infinite linearly independent sets of elements of order two [27][36]. In terms of the n-solvable filtration this implies the existence of

$$\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \subset \mathbb{G}_1.$$

Different $\mathbb{Z}^\infty$-summands were exhibited in [31][16]. More recent work of Se-Goo Kim [29] on the "polynomial splitting" properties of Casson-Gordon invariants led to a generalization analogous to the result of Milnor-Levine:

$$\bigoplus_{p(t)} \mathbb{Z}^\infty \subset \mathbb{G}_1.$$

Thus there is evidence that $\mathbb{G}_1$ also exhibits a $p(t)$-primary decomposition. Further evidence is given in [30]. Although a similar statement for the 2-torsion in $\mathbb{G}_1$ has not appeared, it is expected from combining the work of [29] and Livingston [36]. Several authors have shown that certain knots that projected to classes of order 2 and 4 in $\mathcal{AC}$ are in fact of *infinite order* in $\mathcal{C}$ [37][38][26][20][35]. A number of papers have addressed the non-triviality of $\{\mathbb{G}_n\}$, [18][17][31][16][11][12][13], culminating in [10] where it was shown that, for any integer n, there exists

$$\mathbb{Z}^\infty \subset \mathbb{G}_n.$$

Moreover the recent work [8] of the authors resulted in a generalization of the latter fact, along the lines of the Levine-Milnor primary decomposition and [30]: for each "distinct" n-tuple $\mathcal{P} = (p_1(t), ..., p_n(t))$ of prime polynomials with $p_i(1) = \pm 1$, there is a distinct subgroup $\mathbb{Z}^\infty \cong \mathcal{I}(\mathcal{P}) \subset \mathbb{G}_n$, yielding a subgroup

$$\bigoplus_{\mathbb{P}_n} \mathbb{Z}^\infty \cong \bigoplus_{\mathcal{P}\in\mathbb{P}_n} \mathcal{I}(\mathcal{P}) \subset \mathbb{G}_n. \tag{1.2}$$

Given a knot K, such an n-tuple encodes the orders of certain submodules of the sequence of higher-order Alexander modules of K. Thus one can distinguish concordance classes of knots not only by their classical Alexander polynomials, but also, loosely speaking, *by their higher-order Alexander polynomials*.

Here we show corresponding results for 2-torsion. That is, for any $n \geq 2$, not only will we exhibit

$$\mathbb{Z}_2^\infty \subset \mathbb{G}_n, \tag{1.3}$$

but we also will exhibit, for each "distinct" $(n-1)$-tuple $\mathcal{P} = (p_1(t), ..., p_{n-1}(t))$ of prime polynomials with $p_i(1) = \pm 1$, a distinct subgroup $\mathbb{Z}_2^\infty \cong \mathcal{T}(\mathcal{P}) \subset \mathbb{G}_n$,

$$\bigoplus_{\mathbb{P}_{n-1}} \mathbb{Z}_2^\infty \cong \bigoplus_{\mathcal{P}\in\mathbb{P}_{n-1}} \mathcal{T}(\mathcal{P}) \subset \mathbb{G}_n. \tag{1.4}$$

The representative knots are distinguished by families of von Neumann signature defects associated to their classical Alexander polynomials and "higher-order Alexander polynomials". The precise statement is given in Theorem 5.7. Each of these concordance classes has a negative amphichiral representative that is smoothly slice in a rational homology 4-ball. Thus the classical signatures and the Casson-Gordon signature-defect obstructions [1] (indeed all metabelian obstructions) vanish for these knots [11, Theorem 9.11]. In addition, the s-invariant of Rasmussen [44], the τ-invariant of Ozsváth-Szabó [42], and the δ_{p^n} invariants of Manolescu-Owens and Jabuka [40][25][41] vanish on these concordance classes, since each of these invariants induces a *homomorphism* $\mathcal{C} \to \mathbb{Z}$ and so must have value zero on classes representing torsion in $\mathcal{C}$. Our examples are inspired by those of Livingston, who provided examples that can be used to establish (1.3) in the case $n = 1$ [36]. His examples are distinguished by their Casson-Gordon signature defects. Our examples are distinguished by higher-order $L^{(2)}$-signature defects. It is striking that elements of finite order can sometimes be detected by signatures. The key observation is that, unlike invariants such as the classical knot signatures, the s invariant, the τ-invariant, or the δ-invariants, the invariants arising from higher-order signature defects (including Casson-Gordon invariants) are not additive under connected sum. Therefore there is no reason to expect that they would vanish on elements of finite order.

Our work is further evidence that $\mathbb{G}_n$ exhibits some sort of primary decomposition, but wherein not only the classical Alexander polynomial, but also some higher-order Alexander polynomials are involved.

We remark that [11] also defined a filtration, $\{\mathcal{F}_n^{top}\}$, of the topological concordance group, $\mathcal{C}^{top}$. Since it is known, by work of Freedman and Quinn, that a knot lies in $\mathcal{F}_n^{top}$ if and only if it lies in $\mathcal{F}_n$, all of the results of this paper apply equally well, without change, to the filtration $\{\mathcal{F}_n^{top}\}$. Therefore, for simplicity, in this paper we will always work in the smooth category.

2. The examples

Our examples are inspired by those of Livingston [36], who exhibited an infinite "linearly independent" set of negative amphichiral algebraically slice knots. His examples can be used to establish the existence of the aforementioned

$$\mathbb{Z}_2^\infty \subset \mathbb{G}_1.$$

2.1. **The Building Blocks.** Consider the knot shown on the left-hand side of Figure 2.1. Here J is an arbitrary pure two component string link [32][21]. The disk containing the letter J

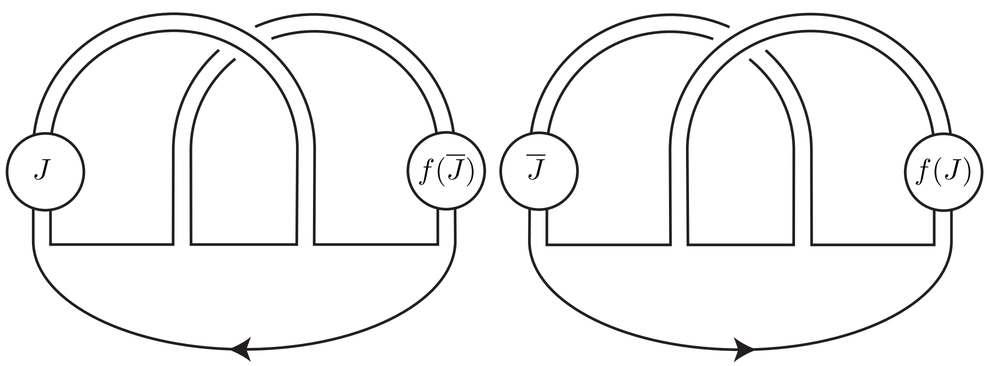

Figure 2.1. Families of Negative Amphichiral Knots K

symbolizes replacing the trivial 2-string link by the 2-string link J. Viewing the knot diagram as being in the xy-plane (y being vertical), the mirror image can be defined as the image under the reflection $(x,y,z) \mapsto (x,y,-z)$, which alters a knot diagram by replacing all positive crossings by negative crossings and vice-versa. Recall that the image of J under this reflection is denoted $\overline{J}$. We also consider a "flip" homeomorphism of S^3 which flips over a diagram, given by rotation of 180 degrees about the y-axis or $f(x,y,z) = (-x,y,-z)$. Note that these homeomorphisms commute. Special cases of the following elementary observation appeared in [36, Lemma 2.1] [35, p.326] and [4, p. 60].

Lemma 2.1. *Suppose J is an arbitrary pure two component string link. Then the knot K on the left-hand side of Figure 2.1 is negative amphichiral.*

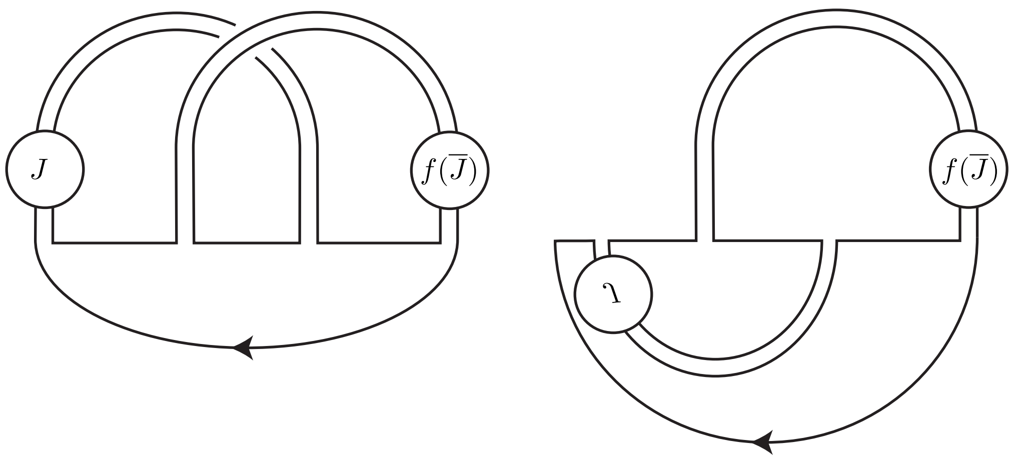

FIGURE 2.2

Proof. The knot on the right-hand side of Figure 2.1 is a diagram for $r(\overline{K})$, since it is obtained by a reflection, in the plane of the paper, of the diagram for K, followed by a reversal of the string orientation. Here we use that f commutes with the reflection. We claim that the result is isotopic to K. Flipping the diagram (rotating by 180 degrees about the vertical axis in the plane of the paper), we arrive at the diagram shown on the left-hand side of Figure 2.2. This is identical to the original diagram of K except that the left-hand band passes under the right-hand band instead of over. But the left-hand band can be "swung" around by an isotopy as suggested in the right-hand side of Figure 2.2, bringing it on top of the other band, at which point one arrives at the original diagram of K. ☐

The following result was first shown for the figure-eight knot (the case that the string link J is a single twist) by the first author (inspired by [14]). It was extended, by Cha, to the case that J is an arbitrary number of twists in [4, p.63]. Our contribution here is just to note that Cha's proof suffices to prove this more general result.

Lemma 2.2. *Each knot K in the family shown in Figure 2.1 is slice in a rational homology 4-ball.*

Proof. We follow the argument of [4], only indicating where our more general argument deviates. It suffices to show that the zero-framed surgery, M_K, as shown on the left-hand side of Figure 2.3, is rational homology cobordant to $S^1 \times S^2$. After adding, to $M_K \times [0,1]$, a four-dimensional 1-handle and 2-handle (going algebraically twice over the 1-handle) and performing certain handle slides (see [4, p.62-64]), one arrives at a 3-manifold M' given by surgery on the 3-component link drawn as the solid lines on the right-hand side of Figure 2.3. Therefore M_K is rationally homology cobordant to M'.

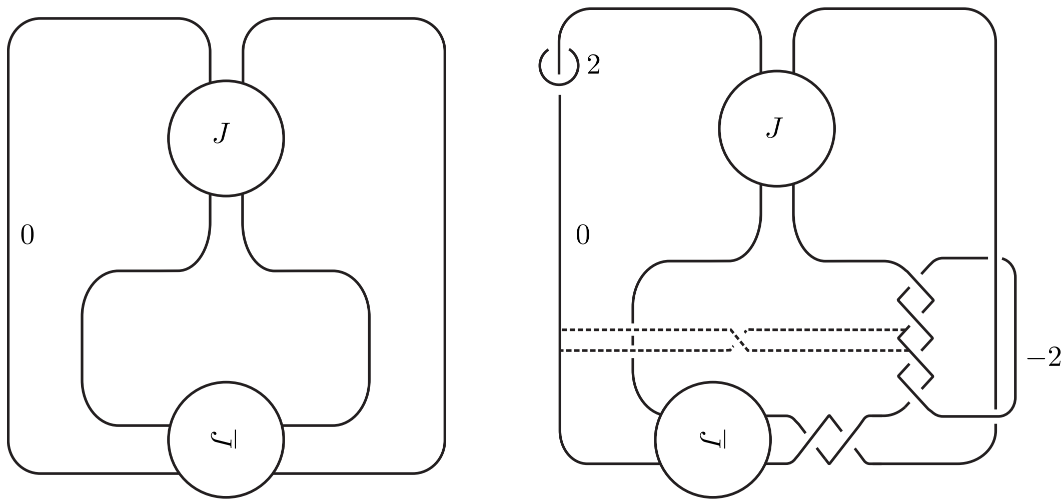

FIGURE 2.3

Next one shows, as follows, that this underlying 3-component link, L_1, is concordant to the simple 3-component link, L_4 shown on the right-hand side of Figure 2.4. Ignoring the framings

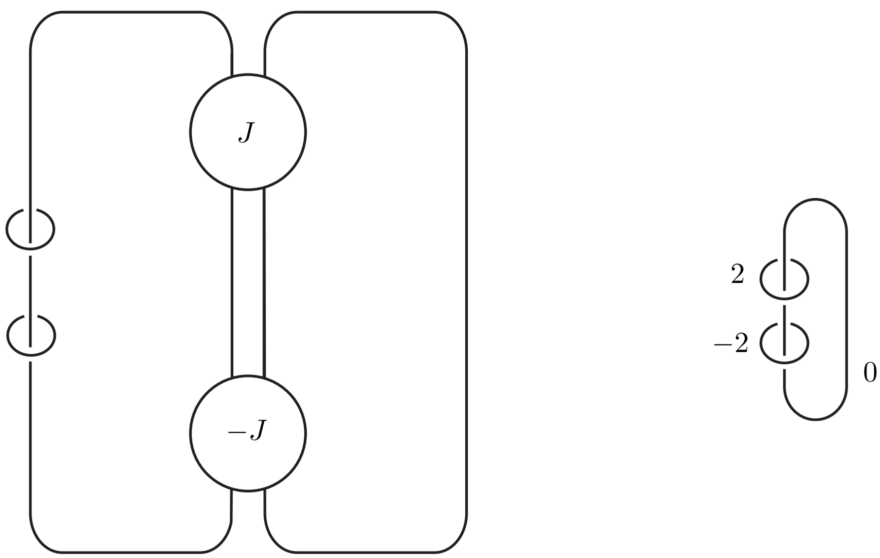

FIGURE 2.4

on L_1, add a band as shown by the dashed lines on the right-hand side of Figure 2.3, resulting in the 4-component link, L_2, shown on the left-hand side of Figure 2.4. Here $-J$ is the image of the (unoriented) string link under the map $(x, y, z) \mapsto (x, -y, z)$. One must be careful here since $\overline{J}$, which is reflection in the plane of the paper, is not the correct notion of mirror image for a string link. Since our y-axis is the true axis of the string link (the $[0,1]$ factor in $D^2 \times [0,1]$), $-J$ is the concordance inverse of J in the string link concordance group [32][22], so $J+(-J)$ is concordant to the trivial 2-string link. Hence the link L_2 is concordant to the 4-component link, L_3, that would result from taking J to be trivial. Capping off the right-most unknotted component of L_3, we arrive at the 3-component link, L_4, shown on the right-hand side of Figure 2.4. This describes the desired concordance from L_1 to L_4. Consequently, M' is homology cobordant to the 3-manifold described by the framed link on the right-hand side of Figure 2.4, which is known to homeomorphic to $S^1 \times S^2$. □

In this paper we will only need the special case of these lemmas wherein the string link J consists of two twisted parallels of a single knotted arc as indicated by the examples in Figures 2.5 and 2.6. Here an m inside the rectangle indicates m full positive twists between the two strands, and the J inside the rectangle indicates that the trivial two component string link has been replaced by two parallel *zero-twisted* copies of a single knotted arc J. This is explained more fully in Subsection 2.2.

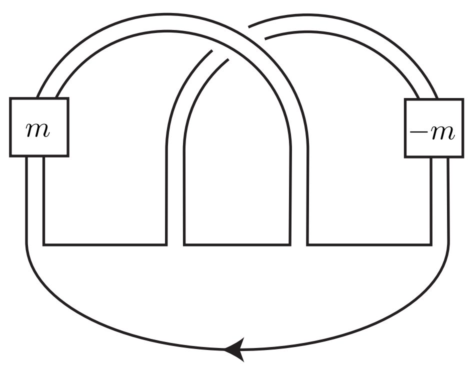

FIGURE 2.5. Negative Amphichiral Knots $\mathbb{E}^m$

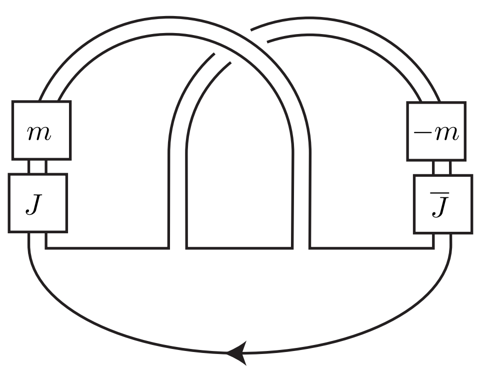

FIGURE 2.6. Families of Negative Amphichiral Knots $\mathbb{E}^m(J)$

Proposition 2.3. *If m and n are distinct positive integers then the Alexander polynomials $\Delta_m(t)$ of $\mathbb{E}^m$ and $\Delta_n(t)$ of $\mathbb{E}^n$ are distinct and irreducible, hence coprime.*

Proof. A Seifert matrix for $\mathbb{E}^m$ with respect to the obvious basis is

$$\begin{pmatrix} m & 0 \\ -1 & -m \end{pmatrix}.$$

Thus the Alexander polynomial of $\mathbb{E}^m$ is $\Delta_m(t) = m^2t^2 - (2m^2+1)t + m^2$. The discriminant $4m^2+1$ is easily seen, for $m \neq 0$, to never be the square of an integer, so the roots of $\Delta_m(t)$ are real and irrational. Hence $\Delta_m(t)$ is irreducible over $\mathbb{Q}[t, t^{-1}]$. It follows that if $\Delta_m(t)$ and $\Delta_n(t)$ had a common factor then they would be identical up to a unit. But the equations $m^2 = qn^2$ and $2m^2+1 = q(2n^2+1)$ imply $q = 1$ so $m = \pm n$. $\square$

2.2. **Doubling Operators.** To construct knots that lie deep in the n-solvable filtration, we use iterated generalized satellite operations.

Suppose R is a knot in S^3 and $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ be an ordered, oriented, trivial link in S^3, that misses R, bounding a collection of oriented disks that meet R transversely as shown on the left-hand side of Figure 2.7. Suppose $(K_1, K_2, \dots, K_m)$ is an m-tuple of auxiliary knots. Let $R_{\vec{\alpha}}(K_1, \dots, K_m)$ denote the result of the operation pictured in Figure 2.7, that is, for each α_j, take the embedded disk in S^3 bounded by α_j; cut off R along the disk; grab the cut strands, tie them into the knot K_j (such that the strands have linking number zero pairwise) and reglue as shown schematically on the right-hand side of Figure 2.7.

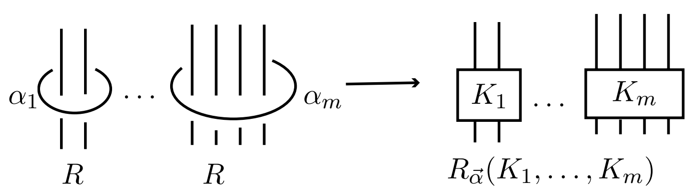

FIGURE 2.7. $R_{\vec{\alpha}}(K_1, \dots, K_m)$: Infection of R by K_j along α_j

We will call this the **result of infection performed on the knot R using the infection knots K_j along the curves α_j** [12]. In the case that $m = 1$ this is the same as the classical satellite construction. This construction has an alternative description. For each α_j, remove a tubular neighborhood of α_j in S^3 and glue in the exterior of a tubular neighborhood of K_j along their common boundary, which is a torus, in such a way that (the longitude of) α_j is identified with the meridian, μ_j, of K_j and the meridian of α_j is identified with the reverse of the longitude, ℓ_j, of K_j as suggested by Figure 2.8. The resulting space can be seen to be homeomorphic to S^3 and the image of R is the new knot.

It is well known that if the input knots K_1 and K_2 are concordant, then the output knots $R_\alpha(K_1)$ and $R_\alpha(K_2)$ are concordant. Thus the functions $R_{\vec{\alpha}}$ descend to $\mathcal{C}$.

Definition 2.4. A **doubling operator**, $R_{\vec{\alpha}} : \mathcal{C} \times \cdots \times \mathcal{C} \to \mathcal{C}$ is a function, as in Figure 2.7, that is given by infection on a ribbon knot R wherein, for each i, $lk(R, \alpha_i) = 0$. Often we suppress α_i from the notation.

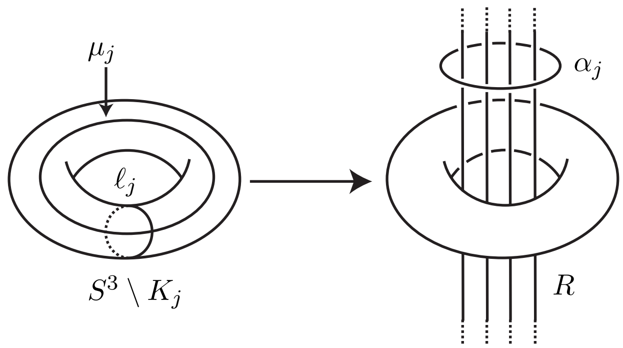

FIGURE 2.8. Infection as replacing a solid torus by a knot exterior

In particular we will consider the family of doubling operators $\mathfrak{R}^m_{\eta_1,\eta_2}(-,-)$ shown in Figure 2.9. Note that, since $\mathbb{E}^m$ is negative amphichiral by Lemma 2.1,

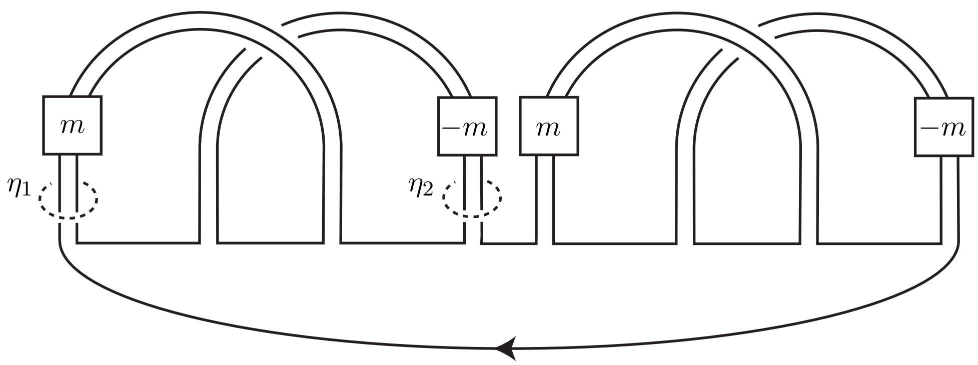

FIGURE 2.9. Negative Amphichiral Doubling Operators $\mathfrak{R}^m \equiv \mathbb{E}^m \# \mathbb{E}^m$

$$\mathfrak{R}^m \equiv \mathbb{E}^m \# \mathbb{E}^m \cong \mathbb{E}^m \# -\mathbb{E}^m,$$

which is well known to be a ribbon knot [45, Exercise 8E.30]. Thus $\mathfrak{R}^m$ is a negative amphichiral ribbon knot. For the case $m = 1$, this was already noted in [36].

We will also consider the family of doubling operators, $\mathcal{R}^m_\alpha$, shown in Figure 2.10 (where here the $-m$ inside a box symbolizes m full negative twists between the bands but where the individual bands remain untwisted), equipped with a specified circle α that can be shown to generate its Alexander module.

2.3. **Elements of order 2 in $\mathcal{F}_n$.** Now we describe large families of examples of negative amphichiral knots that lie in $\mathcal{F}_n$. Let K^0 be any knot with Arf(K^0)$= 0$. Let K^{n-1} be the image of K^0 under the composition of *any* $n-1$ doubling operators (each requiring a single input), that is,

$$K^{n-1} \equiv R^{n-1} \circ \cdots \circ R^1(K^0). \tag{2.1}$$

$\mathcal{R}^m$

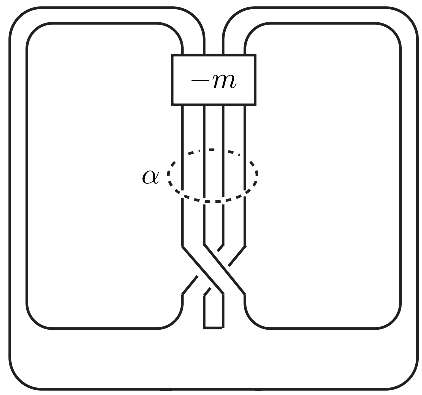

FIGURE 2.10. Doubling operators $\mathcal{R}^m_\alpha$

Then, for any integer m we define K^n as in Figure 2.11, that is, $K^n \equiv \mathfrak{R}^m_{\eta_1,\eta_2}(K^{n-1}, \overline{K^{n-1}})$.

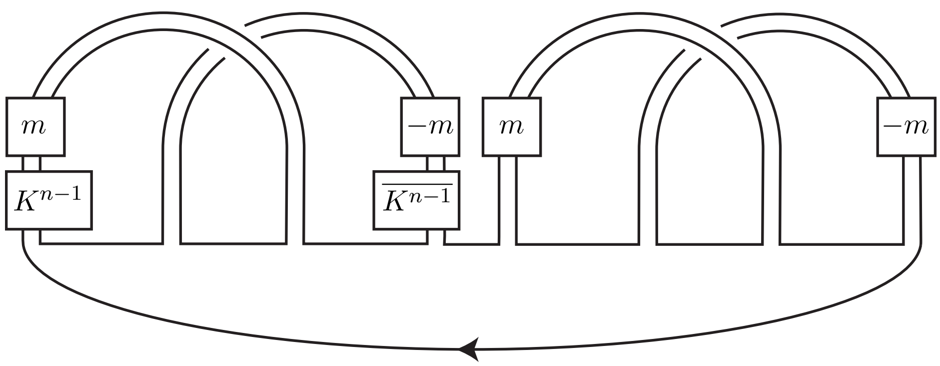

FIGURE 2.11. The examples K^n

Proposition 2.5. *For any $n \geq 1$, any m, any composition of $n-1$ doubling operators and any Arf invariant zero input knot K^0, the knot K^n of Figure 2.11 satisfies*

- *K^n is negative amphichiral;*
- *$K^n \in \mathcal{F}_n$;*
- *K^n is (smoothly) slice in a smooth rational homology 4-ball; and*
- *$K^n \# K^n$ is a slice knot.*

Proof. It was shown in [10, Theorem 7.1] that, for any any doubling operator R,

$$R(\mathcal{F}_i, \ldots, \mathcal{F}_i) \subset \mathcal{F}_{i+1}.$$

Since any knot of Arf invariant zero is known to lie in $\mathcal{F}_0$ [11, Remark 8.14,Thm. 8.11], and since K^n is the image of K^0 under a composition of n doubling operators, it follows that $K^n \in \mathcal{F}_n$.

Note that K^n is the connected sum of two knots each of which is of the form shown in Figure 2.6 (hence of the form of Figure 2.1). Thus, by Lemma 2.2, each such K^n is slice in a rational homology 4-ball. Moreover, by Lemma 2.1, K^n is negative amphichiral so $K^n\#K^n$ is isotopic to $K^n\#r(\overline{K^n})$. But the latter is a ribbon knot and hence a slice knot. $\square$

For specificity we define the following infinite families:

Definition 2.6. Given an n-tuple $(m_1,...,m_n)$ of integers and an Arf invariant zero knot K^0, we define $\mathcal{K}^n(m_1,\ldots,m_n,K^0)$ to be the image of K^0 under the following composition of n doubling operators. Specifically let

$$\mathcal{K}^n \equiv \mathcal{K}^n(m_1,\ldots,m_n,K^0) \equiv \mathfrak{R}^{m_n}_{\eta_1,\eta_2}(\mathcal{K}^{n-1},\overline{\mathcal{K}^{n-1}}),$$

as shown in Figure 2.11, where $\mathcal{K}^{n-1}$ is

$$\mathcal{R}^{m_{n-1}}\circ\cdots\circ\mathcal{R}^{m_1}(K^0),$$

where the $\mathcal{R}^{m_i}$ are the operators of Figure 2.10. In other words, we recursively set:

$$\begin{aligned}
\mathcal{K}^1 &= \mathcal{R}^{m_1}_{\alpha}(K^0);\\
\mathcal{K}^2 &= \mathcal{R}^{m_2}_{\alpha}\circ\mathcal{R}^{m_1}_{\alpha}(K^0);\\
&\vdots\\
\mathcal{K}^{n-1} &= \mathcal{R}^{m_{n-1}}_{\alpha}\circ\cdots\circ\mathcal{R}^{m_1}_{\alpha}(K^0);\\
\mathcal{K}^n &= \mathfrak{R}^{m_n}_{\eta_1,\eta_2}(\mathcal{K}^{n-1},\overline{\mathcal{K}^{n-1}}).
\end{aligned}$$

Even though $\mathcal{K}^n$ depends on $(m_1,...,m_n,K^0)$, we will often suppress the latter from the notation.

3. Commutator Series and Filtrations of the knot concordance groups

To accomplish our goals, we must establish that many of the knots in the families given by Figure 2.11, and specifically those in Definition 2.6, are not in $\mathcal{F}_{n.5}$ and, indeed, are distinct from each other in $\mathcal{F}_n/\mathcal{F}_{n.5}$. To this end we review recent work of authors that introduced refinements of the n-solvable filtration parameterized by certain classes of group series that generalized the derived series. In particular the authors defined specific filtrations of $\mathcal{C}$ that depend on a sequence of polynomials. These filtrations can then be used to distinguish between knots with different Alexander modules or different higher-order Alexander modules. All of the material in this section is a review of the relevant terminology of [8, Sections 2,3].

Recall that the **derived series**, $\{G^{(n)} \mid n\geq 0\}$, of a group G is defined recursively by $G^{(0)}\equiv G$ and $G^{(n+1)}\equiv[G^{(n)},G^{(n)}]$. The **rational derived series** [23], $\{G^{(n)}_r \mid n\geq 0\}$, is defined by $G^{(0)}_r\equiv G$ and

$$G^{(n+1)}_r = \ker\left(G^{(n)}_r \to \frac{G^{(n)}_r}{[G^{(n)}_r,G^{(n)}_r]} \to \frac{G^{(n)}_r}{[G^{(n)}_r,G^{(n)}_r]}\otimes_{\mathbb{Z}}\mathbb{Q}\right).$$

More generally,

Definition 3.1 ([8, Definition 2.1])**.** A **commutator series** defined on a class of groups is a function, $*$, that assigns to each group G in the class a nested sequence of normal subgroups

$$\cdots \lhd G_*^{(n+1)} \lhd G_*^{(n)} \lhd \cdots \lhd G_*^{(0)} \equiv G,$$

such that $G_*^{(n)}/G_*^{(n+1)}$ is a torsion-free abelian group.

Proposition 3.2 ([8, Proposition 2.2])**.** *For any commutator series $\{G_*^{(n)}\}$,*

1. $\{x \in G_*^{(n)} \mid \exists k > 0,\ x^k \in [G_*^{(n)}, G_*^{(n)}]\} \subset G_*^{(n+1)}$ *(and in particular $[G_*^{(n)}, G_*^{(n)}] \subset G_*^{(n+1)}$, whence the name commutator series);*
2. $G_r^{(n)} \subset G_*^{(n)}$*, that is, every commutator series contains the rational derived series;*
3. $G/G_*^{(n)}$ *is a poly-(torsion-free abelian) group (abbreviated PTFA);*
4. $\mathbb{Z}[G/G_*^{(n)}]$ *and* $\mathbb{Q}[G/G_*^{(n)}]$ *are right (and left) Ore domains.*

Any commutator series that satisfies a weak functoriality condition induces a filtration, $\{\mathcal{F}_n^*\}$, of $\mathcal{C}$ by subgroups. These filtrations generalize and refine the (n)-solvable filtration $\{\mathcal{F}_n\}$ of [11]. Let M_K denote the closed 3-manifold obtained by zero-framed surgery on S^3 along K.

Definition 3.3 ([8, Definition 2.3])**.** A knot K is an element of $\boldsymbol{\mathcal{F}_n^*}$ if the zero-framed surgery M_K bounds a compact, connected, oriented, smooth 4-manifold W such that

1. $H_1(M_K;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism;
2. $H_2(W;\mathbb{Z})$ has a basis consisting of connected, compact, oriented surfaces, $\{L_i, D_i \mid 1 \leq i \leq r\}$, embedded in W with trivial normal bundles, wherein the surfaces are pairwise disjoint except that, for each i, L_i intersects D_i transversely once with positive sign;
3. for each i, $\pi_1(L_i) \subset \pi_1(W)_*^{(n)}$ and $\pi_1(D_i) \subset \pi_1(W)_*^{(n)}$.

A knot $K \in \boldsymbol{\mathcal{F}_{n.5}^*}$ if in addition,

4. for each i, $\pi_1(L_i) \subset \pi_1(W)_*^{(n+1)}$.

Such a 4-manifold is called an $\boldsymbol{(n,*)}$**-solution** (respectively an $(n.5,*)$-solution) for K and it is said that K is $\boldsymbol{(n,*)}$**-solvable** (respectively $(n.5,*)$-solvable) via W. The case where the commutator series is the derived series (without the torsion-free abelian restriction) is denoted $\mathcal{F}_n$ and we speak of W being an (n)-solution, and K or M_K being (n)-solvable via W [11, Section 8].

Definition 3.4. A commutator series $\{G_*^{(n)}\}$ is **weakly functorial** (on a class of {groups, maps}) if $f(G_*^{(n)}) \subset \pi_*^{(n)}$ for each n and for any homomorphism $f: G \to \pi$ (in the class) that induces an isomorphism $G/G_r^{(1)} \cong \pi/\pi_r^{(1)}$ (i.e. induces an isomorphism on $H_1(-;\mathbb{Q})$).

Proposition 3.5 ([8, Prop. 2.5])**.** *Suppose $*$ is a weakly functorial commutator series defined on the class of groups with $\beta_1 = 1$. Then $\{\mathcal{F}_n^*\}_{n\geq 0}$ is a filtration by subgroups of the classical (smooth) knot concordance group $\mathcal{C}$:*

$$\cdots \subset \mathcal{F}_{n+1}^* \subset \mathcal{F}_{n.5}^* \subset \mathcal{F}_n^* \subset \cdots \subset \mathcal{F}_1^* \subset \mathcal{F}_{0.5}^* \subset \mathcal{F}_0^* \subset \mathcal{C}.$$

Moreover, for any $n \in \frac{1}{2}\mathbb{Z}$

$$\mathcal{F}_n \subset \mathcal{F}_n^*.$$

The case where the commutator series is the derived series (without the torsion-free abelian restriction) is the **(n)-solvable filtration** [11], denoted $\{\mathcal{F}_n\}$.

3.1. **The Derived Series Localized at $\mathcal{P}$.**

Fix an n-tuple $\mathcal{P} = (p_1(t), ..., p_n(t))$ of non-zero elements of $\mathbb{Q}[t, t^{-1}]$, such that $p_1(t) \doteq p_1(t^{-1})$. For each such $\mathcal{P}$ we now recall from [8] the definition of a partial commutator series that we call the *(polarized) derived series localized at* $\mathcal{P}$, that is defined on the class of groups with $\beta_1 = 1$.

Suppose G is a group such that $G/G_r^{(1)} \cong \mathbb{Z} = \langle \mu \rangle$. Then we define the derived series localized at $\mathcal{P}$ recursively in terms of certain right divisor sets $S_{p_n} \subset \mathbb{Q}[G/G_{\mathcal{P}}^{(n)}]$.

Definition 3.6. For $n \geq 0$, let

$$G_{\mathcal{P}}^{(0)} \equiv G;$$
$$G_{\mathcal{P}}^{(1)} \equiv G_r^{(1)};$$

and for $n \geq 1$

$$G_{\mathcal{P}}^{(n+1)} \equiv \ker\left(G_{\mathcal{P}}^{(n)} \to \frac{G_{\mathcal{P}}^{(n)}}{[G_{\mathcal{P}}^{(n)}, G_{\mathcal{P}}^{(n)}]} \otimes_{\mathbb{Z}[G/G_{\mathcal{P}}^{(n)}]} \mathbb{Q}[G/G_{\mathcal{P}}^{(n)}] S_{p_n}^{-1} \right). \tag{3.1}$$

To make sense of (3.1) one must realize that, for any $H \lhd G$, $H/[H,H]$ is a right $\mathbb{Z}[G/H]$-module where g acts on h by $h \mapsto g^{-1}hg$. One must also verify, at each stage, that $G_{\mathcal{P}}^{(n)}$ has been defined in such a way that $G_{\mathcal{P}}^{(k)}/G_{\mathcal{P}}^{(k+1)}$ is a torsion-free abelian group for each $k < n$, so $G/G_{\mathcal{P}}^{(n)}$ is a poly-(torsion-free-abelian) group (PTFA), from which it follows that $\mathbb{Q}[G/G_{\mathcal{P}}^{(n)}]$ is a right Ore domain. Therefore, for any right divisor set $S_{p_n} \subset \mathbb{Q}[G/G_{\mathcal{P}}^{(n)}]$ we may define the Ore localization $\mathbb{Q}[G/G_{\mathcal{P}}^{(n)}]S_{p_n}^{-1}$ as in (3.1) (see [8, Sections 3,4]).

For the (polarized) derived series localized at $\mathcal{P}$ we use the following right divisor sets:

Definition 3.7. The **(polarized) derived series localized at** $\mathcal{P}$ is defined as in Definition 3.6 by setting

$$S_{p_1} = S_{p_1}(G) = \{q_1(\mu)...q_r(\mu) \mid (p_1(t), q_j(t)) = 1;\; G/G_r^{(1)} \cong \langle \mu \rangle\} \subset \mathbb{Q}[G/G_r^{(1)}]; \tag{3.2}$$

and for $n \geq 2$

$$S_{p_n} = S_{p_n}(G) = \{q_1(a_1)...q_r(a_r) \mid \widetilde{(p_n, q_j)} = 1;\; q_j(1) \neq 0;\; a_j \in G_{\mathcal{P}}^{(n-1)}/G_{\mathcal{P}}^{(n)}\}, \tag{3.3}$$

so $S_{p_n} \subset \mathbb{Q}[G_{\mathcal{P}}^{(n-1)}/G_{\mathcal{P}}^{(n)}]$.

Here $p_i(t)$ and $q_j(t)$ are in $\mathbb{Q}[t, t^{-1}]$. By $(p_1, q_j) = 1$ we mean that p_1 is coprime to q_j in $\mathbb{Q}[t, t^{-1}]$, as usual. But by $\widetilde{(p_n, q_j)} = 1$ we mean something slightly stronger.

Definition 3.8 ([8, Defn. 4.4])**.** Two non-zero polynomials $p(t), q(t) \in \mathbb{Q}[t, t^{-1}]$ are said to be **strongly coprime**, denoted $\widetilde{(p,q)} = 1$ if, for every pair of non-zero integers, n, k, $p(t^n)$ is relatively prime to $q(t^k)$. Alternatively, $\widetilde{(p,q)} \neq 1$ if and only if there exist *non-zero* roots, $r_p, r_q \in \mathbb{C}^*$, of $p(t)$ and $q(t)$ respectively, and non-zero integers k, n, such that $r_p^k = r_q^n$. Clearly, $\widetilde{(p,q)} = 1$ if and only if for each prime factor $p_i(t)$ of $p(t)$ and $q_j(t)$ of $q(t)$, $\widetilde{(p_i, q_j)} = 1$.

Note that $\mathbb{Q} - \{0\} \subset S_{p_n}$ (take q_j to be a non-zero constant). It is easy to see (and was proved in [8, Section 4]) that S_{p_n} is closed (up to units) under the involution on $\mathbb{Q}[G/G^{(n)}_{\mathcal{P}}]$. Here we need $p_1(t) \doteq p_1(t^{-1})$.

Example 3.9. Consider the family of quadratic polynomials

$$\{q_m(t) = (mt - (m+1))((m+1)t - m) \mid m \in \mathbb{Z}^+\},$$

whose roots are $\{m/(m+1), (m+1)/m\}$. The polynomial q_m is the Alexander polynomial of the ribbon knot $\mathcal{R}^m$ shown in in Figure 2.10. It can easily be seen (and was proved in [8, Example 4.10]) that $\widetilde{(q_m, q_n)} = 1$ if $m \neq n$.

Theorem 3.10 ([8, Thm. 4.16])**.** *The (polarized) derived series localized at $\mathcal{P}$ is a weakly functorial commutator series on the class of groups with $\beta_1 = 1$.*

4. von Neumann signature defects as obstructions to $(n.5, *)$-solvability

To each commutator series there exist signature defects that offer obstructions to a given knot lying in a term of $\mathcal{F}^*$. Given a closed, oriented 3-manifold M, a discrete group Γ, and a representation $\phi : \pi_1(M) \to \Gamma$, the **von Neumann ρ-invariant**, $\rho(M, \phi) \in \mathbb{R}$, was defined by Cheeger and Gromov [5]. If $(M, \phi) = \partial(W, \psi)$ for some compact, oriented 4-manifold W and $\psi : \pi_1(W) \to \Gamma$, then it is known that $\rho(M, \phi) = \sigma^{(2)}_\Gamma(W, \psi) - \sigma(W)$ where $\sigma^{(2)}_\Gamma(W, \psi)$ is the $L^{(2)}$-signature (von Neumann signature) of the equivariant intersection form defined on $H_2(W; \mathbb{Z}\Gamma)$ twisted by ψ, and $\sigma(W)$ is the ordinary signature of W [39][13, Section 2]. Thus the ρ-invariants should be thought of as *signature defects*. They were first used to detect non-slice knots in [11]. For a more thorough discussion see [11, Section 5][13, Section 2][12, Section 2]. All of the coefficient systems Γ in this paper will be of the form $\pi/\pi^{(n)}_*$ where π is the fundamental group of a space. Hence all such Γ will be PTFA. Aside from the definition, the properties that we use in this paper are:

Proposition 4.1.

1. *If ϕ factors through $\phi' : \pi_1(M) \to \Gamma'$ where Γ' is a subgroup of Γ, then $\rho(M, \phi') = \rho(M, \phi)$.*
2. *If ϕ is trivial (the zero map), then $\rho(M, \phi) = 0$.*
3. *If $M = M_K$ is the zero-surgery on a knot K and $\phi : \pi_1(M) \to \mathbb{Z}$ is the abelianization, then $\rho(M, \phi)$ is denoted **$\rho_0(K)$** and is equal to the integral over the circle of the Levine-Tristram signature function of K* [12, Prop. 5.1]*. Thus $\rho_0(K)$ is the average of the classical signatures of K.*
4. *If K is a slice knot or link and $\phi : M_K \to \Gamma$ (Γ PTFA) extends over π_1 of a slice disk exterior then $\rho(M_K, \phi) = 0$ by* [11, Theorem 4.2].
5. *The von Neumann signature satisfies Novikov additivity, i.e. if W_1 and W_2 intersect along a common boundary component then $\sigma^{(2)}_\Gamma(W_1 \cup W_2) = \sigma^{(2)}_\Gamma(W_1) + \sigma^{(2)}_\Gamma(W_2)$* [11, Lemma 5.9].

6. *For any* 3*-manifold* M*, there is a positive real number* C_M*, called the* ***Cheeger-Gromov constant*** [5][13, Section 2] *of* M *such that, for* any ϕ
$$|\rho(M,\phi)| < C_M.$$

We will also need the following generalization of property (4).

Theorem 4.2 ([8, Theorem 5.2])**.** *Suppose* $*$ *is a commutator series (no functoriality is required). Suppose* $K \in \mathcal{F}^*_{n.5}$*, so the zero-framed surgery* M_K *is* $(n.5,*)$*-solvable via* W *as in Definition 3.3. Let* $G = \pi_1(W)$ *and consider*
$$\phi : \pi_1(M_K) \to G \to G/G_*^{(n+1)} \to \Gamma,$$
where Γ *is an arbitrary PTFA group. Then*
$$\sigma^{(2)}_\Gamma(W,\phi) - \sigma(W) = 0 = \rho(M_K,\phi).$$

5. Statements of Main Results and the outline of the proof

We will show that for any $n \geq 2$, not only does there exist
$$\mathbb{Z}_2^\infty \subset \mathbb{G}_n \equiv \mathcal{F}_n/\mathcal{F}_{n.5},$$
but there are also many distinct such classes
$$\bigoplus_{\mathbb{P}_{n-1}} \mathbb{Z}_2^\infty \subset \mathbb{G}_n,$$
distinguished by the sequence of orders of certain higher-order Alexander modules of the knots.

Definition 5.1. Given $\mathcal{P} = (p_1(t), ..., p_n(t))$ and $\mathcal{Q} = (q_1(t), ..., q_n(t))$, we say that $\mathcal{P}$ is **strongly coprime** to $\mathcal{Q}$ if either $(p_1, q_1) = 1$, or for some $k > 1$, $(p_k, q_k) = 1$.

Theorem 5.2 ([8, Theorem 6.5])**.** *For any* $n \geq 1$*, let* $R^{n-1}_{\alpha_{n-1}}, \dots, R^1_{\alpha_1}$ *be any doubling operators and* K^0 *be any Arf invariant zero input knot. Consider the knot* $K^n \equiv \mathfrak{R}^m_{\eta_1,\eta_2}(K^{n-1}, \overline{K^{n-1}})$*, where* $K^{n-1} = R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^1_{\alpha_1}(K^0)$*. Then*
$$K^n \in \mathcal{F}^{\mathcal{P}}_{n+1}$$
for each $\mathcal{P} = (p_1(t), p_2(t), ..., p_n(t))$*, with* $p_1(t) \doteq p_1(t^{-1})$*, that is strongly coprime to* $(\Delta_m(t), q_{n-1}(t), \dots, q_1(t))$*, where* Δ_m *is the Alexander polynomial of* $\mathbb{E}^m$ *and* q_i *is the Alexander polynomial of* R^i*.*

This applies, in particular, to the families of Definition 2.6, constructed using the ribbon knots of Figures 2.9 and 2.10.

Corollary 5.3. *For any* $(m_1, \dots, m_n)$ *and any input knot* K^0 *with Arf invariant zero,*
$$\mathcal{K}^n(m_1, \dots, m_n, K^0) \in \mathcal{F}^{\mathcal{P}}_{n+1}$$
for each $\mathcal{P} = (p_1(t), p_2(t), ..., p_n(t))$ *that is strongly coprime to* $(\Delta_{m_n}(t), q_{n-1}(t) \dots, q_1(t))$ *where* Δ_{m_n} *is the Alexander polynomial of* $\mathbb{E}^{m_n}$ *and* q_i *is the Alexander polynomial of* $\mathcal{R}^{m_i}$*.*

Now we need a non-triviality theorem to complement Theorem 5.2.

Theorem 5.4. *Suppose*

$$K^n \equiv \mathfrak{R}^m_{\eta_1,\eta_2}(K^{n-1}, \overline{K^{n-1}}),$$

where K^{n-1} is the result of applying any sequence of $n-1$ doubling operators, $R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^1_{\alpha_1}$ to an Arf invariant zero "input" knot K^0. Suppose additionally that $n \geq 2$ and

1. *$m \neq 0$;*
2. *for each i, α_i generates the rational Alexander module of R^i, and this module is nontrivial;*
3. *$|\rho_0(K^0)|$, the average Levine-Tristram signature of K^0, is greater than twice the sum of the Cheeger-Gromov constants of the ribbon knots $\mathfrak{R}^m$, $R^1, \ldots, R^{n-1}$ (see Section 4).*

If $\mathcal{P}$ is the sequence of classical Alexander polynomials of the knots $(\mathbb{E}^m, R^{n-1}, \ldots, R^1)$, then

$$K^n \notin \mathcal{F}^{\mathcal{P}}_{n.5}.$$

This can be applied to the specific families of Definition 2.6.

Corollary 5.5. *Fix $n \geq 2$ and an n-tuple of positive integers $(m_1, \ldots, m_n)$. Suppose K^0 is chosen so that $|\rho_0(K^0)|$ is greater than twice the sum of the Cheeger-Gromov constants of the ribbon knots $\mathfrak{R}^{m_n}$, $\mathcal{R}^{m_{n-1}}, \ldots, \mathcal{R}^{m_1}$. If $\mathcal{P}$ is the n-tuple of Alexander polynomials of the knots $(\mathbb{E}^{m_n}, \mathcal{R}^{m_{n-1}}, \ldots, \mathcal{R}^{m_1})$, then*

$$\mathcal{K}^n \notin \mathcal{F}^{\mathcal{P}}_{n.5}.$$

The proofs of Theorems 5.2 and 5.4 will constitute Sections 6 and 7. Assuming these theorems, we now derive our main results.

Theorem 5.6. *Fix $n \geq 2$. For any n-tuple of positive integers $(m_1, ..., m_n)$ choose an Arf invariant zero knot $K^0(m_1, ..., m_n)$ such that $|\rho_0(K^0)|$ is greater than twice the sum of the Cheeger-Gromov constants of $\mathfrak{R}^{m_n}$, $\mathcal{R}^{m_{n-1}}, \ldots, \mathcal{R}^{m_1}$. Then the resulting set of knots*

$$\{\mathcal{K}^n(m_1, ..., m_n, K^0) \mid m_i \in \mathbb{Z}^+\},$$

as in Definition 2.6, represent linearly independent, order two elements of $\mathcal{F}_n/\mathcal{F}_{n.5}$. They also represent linearly independent order two elements in $\mathcal{C}$. In particular this gives

$$\mathbb{Z}_2^\infty \subset \mathbb{G}_n \equiv \frac{\mathcal{F}_n}{\mathcal{F}_{n.5}},$$

where each class is represented by a negative amphichiral knot that is slice in a rational homology 4-ball.

Proof of Theorem 5.6 assuming Theorems 5.2 and 5.4. By Proposition 2.5, $\mathcal{K}^n$ is negative amphichiral, $\mathcal{K}^n \in \mathcal{F}_n$ and $\mathcal{K}^n \# \mathcal{K}^n$ is a slice knot. Thus $2[\mathcal{K}^n] = 0$ in $\mathcal{F}_n/\mathcal{F}_{n.5}$. By Corollary 5.5, for a certain $\mathcal{P}$, $\mathcal{K}^n \notin \mathcal{F}^{\mathcal{P}}_{n.5}$, so in particular $\mathcal{K}^n \notin \mathcal{F}_{n.5}$ by Proposition 3.5. Therefore each $[\mathcal{K}^n]$ has order precisely two in $\mathbb{G}_n$.

Suppose there exists a nontrivial relation

$$J = \mathcal{K}^n(m_{11}, ..., m_{1n}, K^0_1) \# ... \# \mathcal{K}^n(m_{k1}, ..., m_{kn}, K^0_k) \in \mathcal{F}_{n.5}.$$

Set $\mathcal{P} = (p_1, ..., p_n) = (\Delta_{1n}, q_{1n-1}, ..., q_{11})$, the reverse of the sequence of Alexander polynomials of the operators corresponding to the *first summand* of J. For each of the *other* summands of

J, the corresponding n-tuple $(m_{i1},...,m_{in})$ is assumed distinct from $(m_{11},...,m_{1n})$. Therefore, the (reversed) sequence of Alexander polynomials of the operators corresponding to this other summand is strongly coprime to $\mathcal{P}$ by Proposition 2.3 and Example 3.9. Thus, by Theorem 5.2, each summand of J, aside from the first, lies in $\mathcal{F}^{\mathcal{P}}_{n+1}$ and hence in $\mathcal{F}^{\mathcal{P}}_{n.5}$. Since $J \in \mathcal{F}_{n.5}$, $J \in \mathcal{F}^{\mathcal{P}}_{n.5}$, by Proposition 3.5. Since $\mathcal{F}^{\mathcal{P}}_{n.5}$ is a subgroup, it would follow that the first summand of J also lay in $\mathcal{F}^{\mathcal{P}}_{n.5}$, contradicting Corollary 5.5. □

More generally,

Theorem 5.7. *Suppose $n \geq 2$. Let $\mathbb{P}_n$ be any set of n-tuples $\mathcal{P} = (\delta_1(t), \delta_2(t), \ldots, \delta_n(t))$ of prime polynomials $\delta_i(t) \in \mathbb{Z}[t, t^{-1}]$ such that $\delta_i(1) = \pm 1$, $\delta_1(t) = \Delta_m = m^2t^2 - (2m^2+1)t + m^2$ and with the property that, for any distinct $\mathcal{P}$, $\mathcal{P}' \in \mathbb{P}_n$, $\mathcal{P}$ and $\mathcal{P}'$ are strongly coprime. Then there exists a set of negative amphichiral n-solvable knots indexed by $\mathbb{P}_n$ that is linearly independent modulo $\mathcal{F}_{n.5}$, that is, that spans*

$$\bigoplus_{\mathbb{P}_n} \mathbb{Z}_2 \subset \mathbb{G}_n,$$

where the knot corresponding to the sequence $(\delta_1(t), \delta_2(t), \ldots, \delta_n(t))$ admits a sequence of higher-order Alexander modules containing submodules whose orders are determined by the sequence $(\delta_1(t)\delta_1(t^{-1}), \ldots, \delta_n(t)\delta_n(t^{-1}))$ with the classical Alexander polynomial being $\delta_1(t)\delta_1(t^{-1})$. Moreover each class is represented by a negative amphichiral knot that is slice in a rational homology 4-ball.

Proof of Theorem 5.7 assuming Theorems 5.2 and 5.4. By [47], for any prime $\delta(t)$ with $\delta(1) = \pm 1$ there exists a ribbon knot whose Alexander module is cyclic of order $\delta(t)\delta(t^{-1})$. Hence, given $\mathcal{P} = (\delta_1(t), \delta_2(t), \ldots, \delta_n(t))$, choose such ribbon knots $R^{n-1}, \ldots, R^1$ whose Alexander polynomials are $\delta_2(t)\delta_2(t^{-1}), \ldots, \delta_n(t)\delta_n(t^{-1})$ respectively, and choose curves α_i (unknotted in S^3), that generate the Alexander modules of the R^i. Thus doubling operators $R^i_{\alpha_i}$, $1 \leq i \leq n-1$, are defined. Since $\delta_1(t) = \Delta_m = m^2t^2 - (2m^2+1)t + m^2$, there is a ribbon knot, namely $\mathfrak{R}^m = \mathbb{E}^m \# \mathbb{E}^m$ of Figure 2.9, whose Alexander polynomial is $\delta_1(t)\delta_1(t^{-1})$. The hypotheses imply $m \neq 0$. Choose any Arf invariant zero knot K^0 such that $|\rho_0(K^0)|$ is greater than twice the sum of the Cheeger-Gromov constants of $\mathfrak{R}^m$, $R^{n-1}, \ldots, R^1$. Then set

$$K^n_{\mathcal{P}} \equiv \mathfrak{R}^m_{\eta_1,\eta_2}(K^{n-1}, \overline{K^{n-1}}), \tag{5.1}$$

where $K^{n-1} \equiv R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^1_{\alpha_1}(K^0)$. To each $\mathcal{P}$ there is an associated n-tuple, $\mathcal{P}^* = (\delta_1, \delta_2(t)\delta_2(t^{-1}), \ldots, \delta_n(t)\delta_n(t^{-1}))$, that gives the sequence of Alexander polynomials of the knots $\mathbb{E}^m, R^{n-1}, \ldots, R^1$ that define $K^n_{\mathcal{P}}$.

By Lemma 2.1 and Proposition 2.5, each $K^n_{\mathcal{P}}$ is negative amphichiral and n-solvable. By Theorem 5.4,

$$K^n_{\mathcal{P}} \notin \mathcal{F}^{\mathcal{P}^*}_{n.5}, \tag{5.2}$$

so $K^n_{\mathcal{P}} \notin \mathcal{F}_{n.5}$. Thus $[K^n_{\mathcal{P}}]$ has order precisely two in $\mathbb{G}_n$. Suppose there were a non-trivial relation

$$J = \sum_{i=1}^{k} K^n_{\mathcal{P}_i} \in \mathcal{F}_{n.5}.$$

By hypothesis, if $i \neq 1$ then $\mathcal{P}_i$ is strongly coprime to $\mathcal{P}_1$. It follows that $\mathcal{P}^*_i$ is strongly coprime to $\mathcal{P}^*_1$. Thus, by Theorem 5.2, if $i \neq 1$ then

$$K^n_{\mathcal{P}_i} \in \mathcal{F}^{\mathcal{P}^*_1}_{n+1} \subset \mathcal{F}^{\mathcal{P}^*_1}_{n.5}.$$

Since $J \in \mathcal{F}_{n.5}$, $J \in \mathcal{F}^{\mathcal{P}^*_1}_{n.5}$. Since the latter is a subgroup,

$$K^n_{\mathcal{P}_1} \in \mathcal{F}^{\mathcal{P}^*_1}_{n.5},$$

contradicting (5.2).

It remains only to relate the sequence $\mathcal{P}$ to the higher-order Alexander modules of the knots $K^n_{\mathcal{P}}$. Since this is not central to our results, we sketch the proof. Recall:

Definition 5.8 ([6, Def. 2.8][23, Def. 5.3])**.** The i^{th}, $i \geq 1$, **higher-order** (integral) **Alexander module of a knot** K is

$$\mathcal{A}^{\mathbb{Z}}_i(K) \equiv H_1(M_K; \mathbb{Z}[G/G^{(i+1)}_r]) \cong \frac{G^{(i+1)}_r}{[G^{(i+1)}_r, G^{(i+1)}_r]},$$

where $G \equiv \pi_1(M_K)$. Note: The case $i = 0$ would give the classical Alexander module.

Thus $\mathcal{A}^{\mathbb{Z}}_i(K^n_{\mathcal{P}})$ is a module over $\Gamma_i \equiv G/G^{(i+1)}_r$, where $G \equiv \pi_1(M_{K^n_{\mathcal{P}}})$. The following lemma shows that the (two) images of the *classical* Alexander polynomial, $\delta_{i+1}(t)\delta_{i+1}(t^{-1})$, of the constituent operator R^{n-i} under certain maps

$$\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[G^{(i)}/G^{(i+1)}_r] \subset \mathbb{Z}\Gamma_i,$$

wherein $t \mapsto x_1$ and $t \mapsto x_2$, appear as the orders of cyclic submodules of $\mathcal{A}^{\mathbb{Z}}_i(K^n_{\mathcal{P}})$.

Lemma 5.9. *Fixing $\mathcal{P} = (\delta_1(t), \delta_2(t), \dots, \delta_n(t))$, for each $1 \leq i \leq n-1$, the i^{th} higher-order Alexander module of $K^n_{\mathcal{P}}$ (the knot defined in* (5.1)*) contains two non-trivial summands*

$$\frac{\mathbb{Z}\Gamma_i}{\delta_{i+1}(x_1)\delta_{i+1}(x_1^{-1})\mathbb{Z}\Gamma_i} \oplus \frac{\mathbb{Z}\Gamma_i}{\delta_{i+1}(x_2)\delta_{i+1}(x_2^{-1})\mathbb{Z}\Gamma_i}$$

for certain $x_1, x_2 \in G^{(i)}/G^{(i+1)}_r$.

Proof. Recall that $K^n_{\mathcal{P}}$ is defined as the image of K^0 under a composition of n doubling operators. In particular $K^{n-1} \equiv R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^1_{\alpha_1}(K^0)$. Sequences of satellite operations have a certain associativity property yielding, for each $i \geq 2$, an alternative description of K^{n-1} as a *single* infection on single ribbon knot, $\tilde{R}^i$, along a curve lying in $\pi_1(S^3-\tilde{R}^i)^{(i-1)}$, using the knot K^{n-i} [7, Prop. 4.7][9, Prop. 5.10]. Specifically,

$$\begin{aligned}
K^{n-1} &= R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^{n-i+1}_{\alpha_{n-i+1}} \left(R^{n-i}_{\alpha_{n-i}} \cdots \circ R^1_{\alpha_1}(K^0)\right)\\
K^{n-1} &= R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^{n-i+1}_{\alpha_{n-i+1}}(K^{n-i})\\
K^{n-1} &= \left(R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^{n-i+2}_{\alpha_{n-i+2}}(R^{n-i+1}_{\alpha_{n-i+1}})\right)_{\beta_i}(K^{n-i})\\
K^{n-1} &= \tilde{R}^i_{\beta_i}(K^{n-i}),
\end{aligned}$$

where

$$\tilde{R}^i_{\beta_i} \equiv R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^{n-i+2}_{\alpha_{n-i+2}}(R^{n-i+1}_{\alpha_{n-i+1}})$$

and β_i is the image of α_{n-i+1}. The specific nature of $\tilde{R}^i$ is not important to our present considerations. If $i=1$, let $\tilde{R}^i_{\beta_i}$ be the identity operator. Then, for any $i \geq 1$, it follows that

$$K^n = \mathfrak{R}^m_{\eta_1,\eta_2}\left(\tilde{R}^i_{\beta_i}(K^{n-i}), \overline{\tilde{R}^i_{\beta_i}(K^{n-i})}\right).$$

This can be reformulated, by the same considerations as above, to yield

$$K^n = \tilde{\mathfrak{R}}_{\gamma_1,\gamma_2}\left(K^{n-i}, \overline{K^{n-i}}\right)$$

where $\tilde{\mathfrak{R}} = \mathfrak{R}^m_{\eta_1,\eta_2}(\tilde{R}^i, \tilde{R}^i)$ and $\{\gamma_1,\gamma_2\}$ are the images of the two copies of β_i. These curves can inductively shown to lie in $\pi_1(S^3-\tilde{\mathfrak{R}})^{(i)}$ [7, Prop. 4.7][9, Prop. 5.10]. The latter computation is very similar to the computation we will perform in (7.15).

Now we can apply known results about the effect of single infection on the higher-order Alexander modules [33, Theorem 3.5][6, Theorem 8.2]:

$$\mathcal{A}^{\mathbb{Z}}_i(K^n) = \mathcal{A}^{\mathbb{Z}}_i(\tilde{\mathfrak{R}}) \oplus \left(\mathcal{A}^{\mathbb{Z}}_0(K^{n-i}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}\Gamma_i\right) \oplus \left(\mathcal{A}^{\mathbb{Z}}_0(\overline{K}^{n-i}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}\Gamma_i\right).$$

where $\mathcal{A}^{\mathbb{Z}}_0$ denotes the classical Alexander module and the first tensor product is given by $t \mapsto x_1 = \gamma_1$ and the second by $t \mapsto x_2 = \gamma_2$. But

$$\mathcal{A}^{\mathbb{Z}}_0(K^{n-i}) \cong \mathcal{A}^{\mathbb{Z}}_0(R^{n-i}) \cong \frac{\mathbb{Z}[t,t^{-1}]}{\delta_{i+1}(t)\delta^{-1}_{i+1}(t)\mathbb{Z}[t,t^{-1}]}.$$

where $t \mapsto x_1$. The Alexander modules of $\overline{R^{n-i}}$ and R^{n-i} are isomorphic. Thus $\mathcal{A}^{\mathbb{Z}}_i(K^n)$ contains two cyclic summands as claimed. By [33, Theorem 3.5][6, Theorem 8.2] these summands are non-zero precisely when x_1 and x_2 are not zero in Γ_i. The verification of the latter requires further computation as in [7, Theorem 4.11][9, Propoposition 5.14]. These calculations are entirely similar to and easier than the ones we will do to verify our Proposition 7.4. They are not included.

This concludes what we will say about the connections between $\mathcal{P}$ and the orders of the higher-order Alexander modules of $K^n_{\mathcal{P}}$. □

This concludes the proof of Theorem 5.7.

□

6. Sketch of Proof of Theorem 5.2

Theorem 5.2 is a consequence of [8, Theorem 6.5]. However, we shall sketch the proof since the basic idea is elementary and it also shows that $K^n \in \mathcal{F}_n$.

Theorem 5.2 ([8, Theorem 6.5])**.** For any $n \geq 1$ and $m \in \mathbb{Z}$, let $R^{n-1}_{\alpha_{n-1}}, \ldots, R^1_{\alpha_1}$ be any doubling operators and K^0 be any Arf invariant zero input knot. Consider the knot $K^n \equiv \mathfrak{R}^m_{\eta_1,\eta_2}(K^{n-1}, \overline{K^{n-1}})$, where $K^{n-1} = R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^1_{\alpha_1}(K^0)$. Then

$$K^n \in \mathcal{F}^{\mathcal{P}}_{n+1}$$

for each $\mathcal{P} = (p_1(t), p_2(t), ..., p_n(t))$, with $p_1(t) \doteq p_1(t^{-1})$, that is strongly coprime to $(\Delta_m(t), q_{n-1}(t), \ldots, q_1(t))$, where Δ_m is the Alexander polynomial of $\mathbb{E}^m$ and q_i is the Alexander polynomial of R^i.

Proof of Theorem 5.2. We set $K^1 = R^1(K^0), \ldots, K^i = R^i(K^{i-1})$ for $i = 1, \ldots, n-1$ and $K^n = \mathfrak{R}^m(K^{n-1}, \overline{K^{n-1}})$. Recall from [10, Lemma 2.3, Figure 2.1] that, whenever a knot L is obtained from a knot R by infection using knots $K_1, K_2, \ldots$ there is a cobordism E whose boundary is the disjoint union of the zero surgeries M_L, $-M_R$ and $-M_{K_1}$, $-M_{K_2}$ et cetera, as shown on the left-hand side of Figure 6.1. Therefore, since $K^n = \mathfrak{R}^m(K^{n-1}, \overline{K^{n-1}})$, there

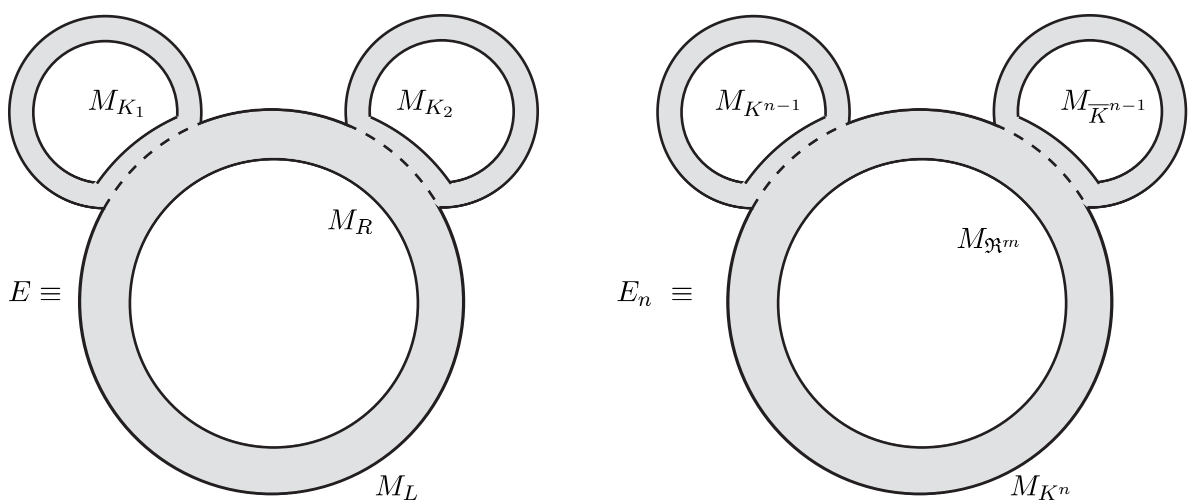

Figure 6.1. The cobordism

is a cobordism E_n whose boundary is the disjoint union of the zero surgeries on K^n, K^{n-1}, $\overline{K^{n-1}}$ and $\mathfrak{R}^m$ as shown on the right-hand side of Figure 6.1 and schematically in Figure 6.2. Similarly there is a cobordism E_i, for $1 \leq i < n$ whose boundary is the disjoint union of the zero surgeries on K^i, K^{i-1} and R^i. Consider $X = E_n \cup E_{n-1} \cup \overline{E}_{n-1} \cup ... \cup E_1 \cup \overline{E}_1$, gluing E_i to E_{i-1} along their common boundary component $M_{K^{i-1}}$, and gluing $\overline{E}_i$ to $\overline{E}_{i-1}$ along their common boundary component $M_{\overline{K^{i-1}}}$, as shown schematically in Figure 6.2. The boundary of X is a disjoint union of M_{K^n}, $-M_{\mathfrak{R}^m}$, $-M_{K^0}$, $-M_{\overline{K^0}}$ and two copies each of $\pm M_{R^{n-1}}, ..., \pm M_{R^1}$. For $1 \leq i < n$, let S_i denote the exterior of any ribbon disk in B^4 for the ribbon knot R^i. Let S_n denote the exterior of any ribbon disk in B^4 for the ribbon knot $\mathfrak{R}^m$. Since Arf(K^0)= 0, $K^0 \in \mathcal{F}_0$ via some V [12, Section 5]. Gluing V, $\overline{V} = -V$ and all the S_i and $\overline{S}_i$ to X, we obtain a 4-manifold, Z as shown in Figure 6.2. Note $\partial Z = M_{K^n}$. We claim that,

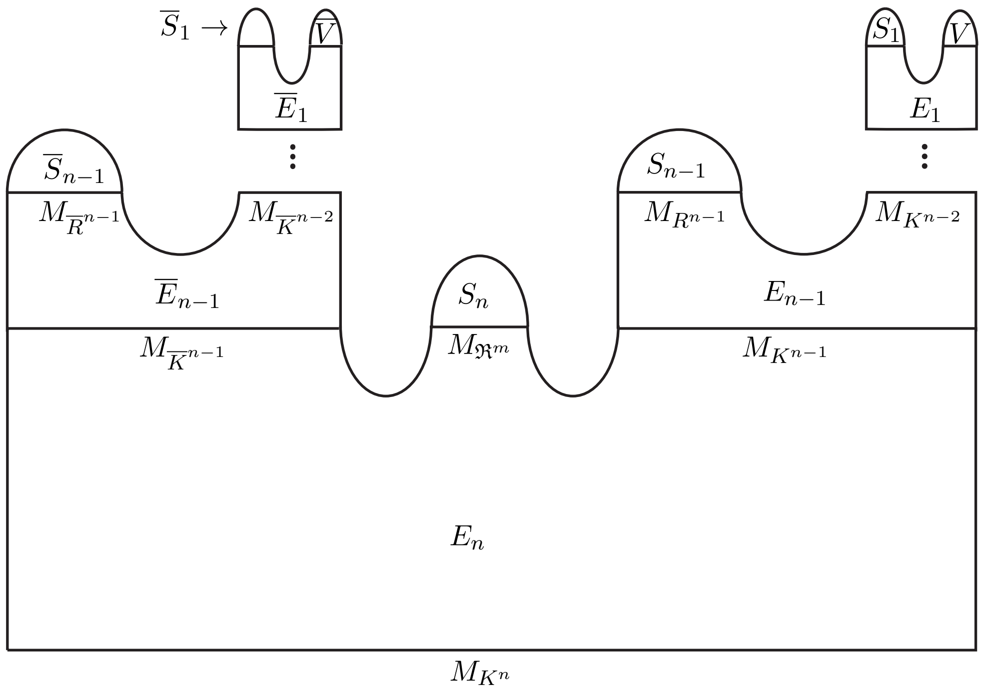

FIGURE 6.2. Z

$$K^n \in \mathcal{F}_n \text{ via } Z, \tag{6.1}$$

and if $\mathcal{P}$ is strongly coprime to $(\Delta_m(t), q_{n-1}(t), \ldots, q_1(t))$, then

$$K^n \in \mathcal{F}^{\mathcal{P}}_{n+1} \text{ via } Z. \tag{6.2}$$

First, simple Mayer-Vietoris sequences together with an analysis of the homology of the E_i (as given by Lemma 6.1 below) imply that $H_2(Z) \cong H_2(V) \oplus H_2(\overline{V})$ since $H_2(S_i) = 0$. Since V is a 0-solution, $H_2(V)$ has a basis of connected compact oriented surfaces, $\{L_j, D_j | 1 \leq j \leq r\}$, satisfying the conditions of Definition 3.3. Similarly for $H_2(\overline{V})$. We claim that,

$$\pi_1(V) \subset \pi_1(Z)^{(n)} \tag{6.3}$$

and if $\mathcal{P}$ is strongly coprime to $(\Delta_m(t), q_{n-1}(t), \ldots, q_1(t))$ then

$$\pi_1(V) \subset \pi_1(Z)^{(n+1)}_{\mathcal{P}}. \tag{6.4}$$

Then,

$$\pi_1(L_j) \subset \pi_1(V) \subset \pi_1(Z)^{(n)},$$

and if $\mathcal{P}$ is strongly coprime to $(\Delta_m(t), q_{n-1}(t), \ldots, q_1(t))$,

$$\pi_1(L_j) \subset \pi_1(V) \subset \pi_1(Z)^{(n+1)}_{\mathcal{P}},$$

and similarly for $\pi_1(D_j)$. The same holds for $\overline{V}$. This would complete the verification of claims (6.1) and (6.2) since $\{L_j, D_j\}$ (together with their counterparts in $\overline{V}$ would then satisfy the criteria of Definition 3.3. The equations (6.3) and (6.4) are then shown inductively in the proof of [8, Theorem 6.2, Theorem 6.5] using the fact that, for each i, the doubling operator $R^i_{\alpha_i}$ satisfies $\ell k(\alpha_i, R^i) = 0$ leading to the fact that

$$\pi_1(M_{K^{i-1}}) \subset \pi_1(E_i)^{(1)}.$$

This concludes our sketch of the proof as given in [8, Theorem 6.5]. We include the relevant result about the elementary topology of the cobordisms E. We will need several of these properties in later proofs.

Lemma 6.1 ([10, Lemma 2.5]). *With regard to E on the left-hand side of Figure 6.1, the inclusion maps induce*

(1) *an epimorphism $\pi_1(M_L) \to \pi_1(E)$ whose kernel is the normal closure of the longitudes of the infecting knots K_i viewed as curves $\ell_i \subset S^3 - K_i \subset M_L$;*
(2) *isomorphisms $H_1(M_L) \to H_1(E)$ and $H_1(M_R) \to H_1(E)$;*
(3) *and isomorphisms $H_2(E) \cong H_2(M_L) \oplus_i H_2(M_{K_i}) \cong H_2(M_R) \oplus_i H_2(M_{K_i})$.*
(4) *The meridian of K, $\mu_K \subset M_K$ is isotopic in E to both $\alpha \subset M_R$ and to the longitudinal push-off of α, often called $\alpha \subset M_L$ by abuse of notation.*
(5) *The longitude of K, $\ell_K \subset M_K$ is isotopic in E to the reverse of the meridian of α, $(\mu_\alpha)^{-1} \subset M_L$ and to the longitude of K in $S^3 - K \subset M_L$ and to the reverse of the meridian of α, $(\mu_\alpha)^{-1} \subset M_R$ (the latter bounds a disk in M_R).*

□

7. Proof of Theorem 5.4

The proof of Theorem 5.4 will occupy the remainder of the paper.

Theorem 5.4. Consider knots K^n, $n \geq 2$ as in Figure 2.11

$$K^n \equiv \mathfrak{R}^m_{\eta_1,\eta_2}(K^{n-1}, \overline{K^{n-1}}),$$

where K^{n-1} is the result of applying a composition of $n-1$ doubling operators, $R^{n-1}_{\alpha_{n-1}} \circ \cdots \circ R^1_{\alpha_1}$ to some Arf invariant zero input knot K^0. Suppose additionally that

1. $m \neq 0$;

2. for each i, α_i generates the rational Alexander module of R^i, and this module is non-trivial;
3. $|\rho_0(K^0)|$, the average Levine-Tristram signature of K^0, is greater than twice the sum of the Cheeger-Gromov constants of the ribbon knots $\mathfrak{R}^m$, $R^1, \dots, R^{n-1}$ (see Section 4).

If $\mathcal{P}$ is the n-tuple of Alexander polynomials of the knots $(\mathbb{E}^m, R^{n-1}, \dots, R^1)$, then

$$K^n \notin \mathcal{F}^{\mathcal{P}}_{n.5}.$$

Proof of Theorem 5.4. We assume that

$$\mathcal{P} = (p_1(t), \dots, p_n(t)) = (\Delta_m, q_{n-1}(t), \dots, q_1(t))$$

is the n-tuple of Alexander polynomials of the knots $(\mathbb{E}^m, R^{n-1}, \dots, R^1)$. Suppose that $K^n \in \mathcal{F}^{\mathcal{P}}_{n.5}$. Let V be the putative $(n.5, \mathcal{P})$-solution. We will derive a contradiction.

Let W_0 be the 4-manifold (refer to Figure 7.1) obtained from V by adjoining the cobordisms E_n, E_{n-1}, $\overline{E}_{n-1}, \dots E_1, \overline{E}_1$ as defined in the proof of Theorem 5.2. For specificity, set

$$\begin{aligned}
&W_n = V,\\
&W_{n-1} = W_n \cup E_n,\\
&W_{n-2} = W_{n-1} \cup E_{n-1} \cup \overline{E}_{n-1},\\
&\vdots\\
&W_0 = W_1 \cup E_1 \cup \overline{E}_1.
\end{aligned}$$

Note that, unlike in the manifold Z of Figure 6.2, we do not cap off the zero surgeries on the various ribbon knots. Thus the boundary of W_0 is the disjoint union of the zero surgeries on the ribbon knots $\mathfrak{R}^m$, $R^{n-1}, \dots, R^1$, $\overline{R}^{n-1}, \dots, \overline{R}^1$, together with the zero surgeries on K^0, $\overline{K}^0$.

Below we will define a commutator series $\{\pi^{(n)}_{\mathcal{S}}\}$ that is slightly larger than the derived series localized at $\mathcal{P}$. In particular,

$$\pi_1(W_0)^{(n+1)}_{\mathcal{P}} \subset \pi_1(W_0)^{(n+1)}_{\mathcal{S}}. \tag{7.1}$$

Then we consider the coefficient system on W_0 given by the projection

$$\phi : \pi_1(W_0) \to \pi_1(W_0)/\pi_1(W_0)^{(n+1)}_{\mathcal{P}} \to \pi_1(W_0)/\pi_1(W_0)^{(n+1)}_{\mathcal{S}}.$$

The bulk of the proof (14 pages!) will be to show that:

(7.2) the restriction of ϕ to $\pi_1(M_{K^0} \subset \partial W_0)$ factors non-trivially through $\mathbb{Z}$; and

(7.3) the restriction of ϕ to $\pi_1(M_{\overline{K}^0} \subset \partial W_0)$ is zero.

We now show that (7.1), (7.2) and (7.3) imply Theorem 5.4. Consider the von Neumann signature defect of (W_0, ϕ):

$$\sigma^{(2)}(W_0, \phi) - \sigma(W_0).$$

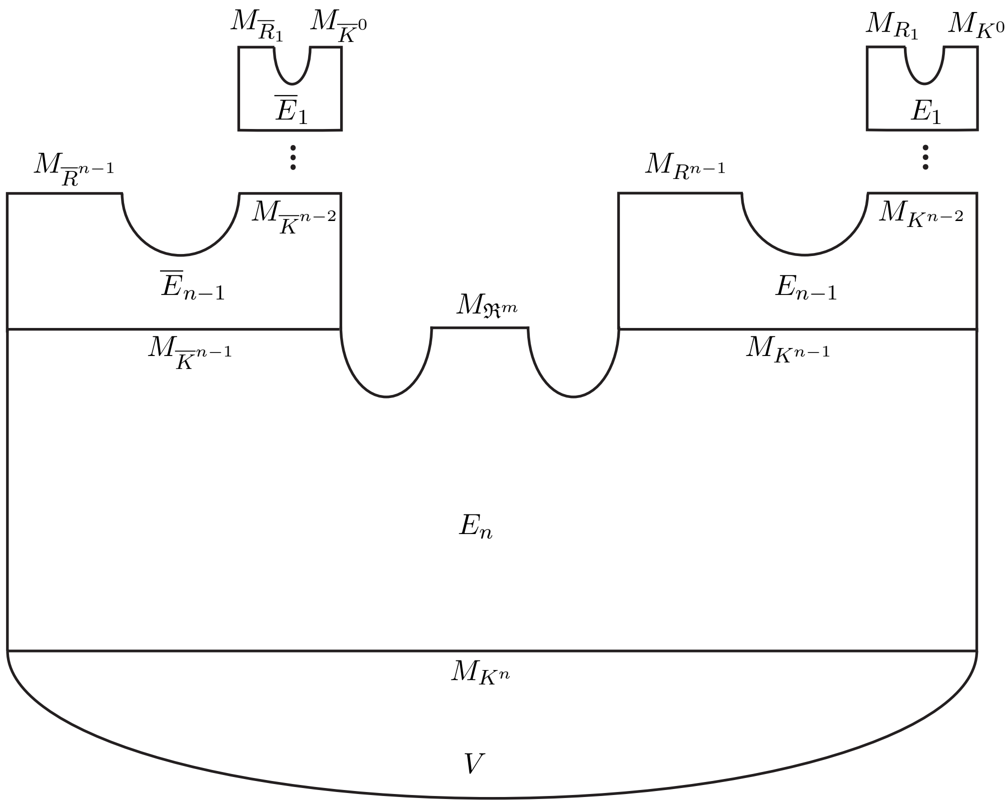

FIGURE 7.1. W_0

By the additivity of these signatures (property (5) of Proposition 4.1), this quantity is the sum of the signature defects for V and those of the E_i and $\overline{E}_i$. Note that the coefficient system on $\pi_1(V)$ factors

$$\pi_1(V) \to \frac{\pi_1(V)}{\pi_1(V)^{(n+1)}_{\mathcal{P}}} \to \frac{\pi_1(W_0)}{\pi_1(W_0)^{(n+1)}_{\mathcal{P}}} \to \frac{\pi_1(W_0)}{\pi_1(W_0)^{(n+1)}_{\mathcal{S}}},$$

where we used Theorem 3.10 to establish the second map and we use (7.1) for the third map. Thus, since V is an $(n.5, \mathcal{P})$-solution, the signature defect of V vanishes by Theorem 4.2. All of the signature defects of the E_i vanish by [10, Lemma 2.4] (essentially because $H_2(E)$ comes from $H_2(\partial E)$). Therefore the signature defect vanishes for W_0. On the other hand, by Section 4,

$$\sigma^{(2)}(W_0, \phi) - \sigma(W_0) = \rho(\partial W_0, \phi).$$

Hence

$$0 = \rho(M_{\mathfrak{R}^m}, \phi) + \cdots + \rho(M_{R^1}, \phi) + \rho(M_{\overline{R^1}}, \phi) + \rho(M_{K^0}, \phi) + \rho(M_{\overline{K^0}}, \phi).$$

By (7.2) and properties (1) and (3) of Proposition 4.1,

$$\rho(M_{K^0}, \phi) = \rho_0(K^0);$$

while by (7.3) and properties (1) and (2) of Proposition 4.1

$$\rho(M_{\overline{K}^0}, \phi) = 0.$$

But, by choice, $|\rho_0(K^0)|$ is greater than the twice the sum of the Cheeger-Gromov constants of the 3-manifolds $M_{\mathfrak{R}^m}, \ldots, M_{R^1}$, which is a contradiction (see property (6) of Proposition 4.1).

Therefore the proof of Theorem 5.4 is reduced to defining a commutator series $\{\pi_{\mathcal{S}}^{(n)}\}$ such that (7.1), (7.2) and (7.3) hold.

The commutator series $\pi_{\mathcal{S}}^{(i)}$ will be defined only for the groups $\pi = \pi_1(W_i)$, because we need not be concerned with any other groups. It will be defined exactly as in Definition 3.6 except that the sequence of right divisor sets $S_1, ..., S_n$ will be slightly different than those of Definition 3.7. We now define $S_1, ..., S_n$. In these definitions π is the fundamental group of one of the W_i.

We define

$$S_1 = S_1(\pi) = S_{p_1} = S_{p_1}(\pi) = \{q_1(\mu)...q_r(\mu) \mid (p_1(t), q_j(t)) = 1;\ \pi/\pi^{(1)} \cong \langle\mu\rangle\}.$$

(Note that $\pi^{(1)} = \pi_r^{(1)} = \pi_{\mathcal{P}}^{(1)} = \pi_{\mathcal{S}}^{(1)}$.) Before defining the other S_i we make a few remarks. Since $p_1(t)$ is a knot polynomial, $p_1(t) \doteq p_1(t^{-1})$, so S_1 is closed (up to units) under the natural involution. In fact, since $p_1(t) = \Delta_m(t)$ is the Alexander polynomial of $\mathbb{E}^m$, $p_1(t)$ is prime. Hence one sees that

$$S_1 = \mathbb{Q}[\mu, \mu^{-1}] - \langle\Delta_m(\mu)\rangle.$$

Therefore for any $\mathbb{Q}[\mu^{\pm 1}]$-module M, $MS_1^{-1} = M_{\langle\Delta_m\rangle}$, the **classical localization of $\boldsymbol{M}$ at the prime ideal $\langle\boldsymbol{\Delta_m}\rangle$**.

Therefore, by (3.1),

$$\pi_{\mathcal{S}}^{(2)} = \pi_{\mathcal{P}}^{(2)} \equiv \ker\left(\pi^{(1)} \to \frac{\pi^{(1)}}{[\pi^{(1)}, \pi^{(1)}]} \otimes \mathbb{Q}[\mu, \mu^{-1}]S_1^{-1} \equiv \mathcal{A}(W)S_1^{-1} \equiv \mathcal{A}(W)_{(\Delta)}\right), \tag{7.4}$$

where $\boldsymbol{\mathcal{A}(W)_{(\Delta)}}$ is the classical localization of $\mathcal{A}(W)$ at the prime $\langle\Delta_m\rangle$. (If W is any space with $\pi_1(W) = \pi$ and $H_1(W) \cong \mathbb{Z}$ then by its integral Alexander module, denoted $\mathcal{A}^{\mathbb{Z}}(W)$ we mean $H_1(W; \mathbb{Z}[\mu, \mu^{-1}]) \cong \pi^{(1)}/\pi^{(2)}$. By its rational Alexander module, denoted $\mathcal{A}(W)$, we mean $H_1(W; \mathbb{Q}[\mu, \mu^{-1}])$.)

Now let $\Gamma \equiv \pi/\pi_{\mathcal{S}}^{(2)} \equiv \pi/\pi_{\mathcal{P}}^{(2)}$ and $A = \pi^{(1)}/\pi_{\mathcal{S}}^{(2)} \equiv \pi^{(1)}/\pi_{\mathcal{P}}^{(2)} \subset \Gamma$. Thus Γ is the semidirect product of the abelian group A with $\pi/\pi^{(1)} \cong \mathbb{Z}$. Note that the circle η_2 (see Figure 2.9) represents an element of $\pi_1(M_{K^n})^{(1)}$ and hence, under inclusion, an element of $\pi^{(1)}$ for each of the groups $\pi = \pi_1(W_i)$ under consideration. Hence, for any π, η_2 has an unambiguous interpretation as an element of A. By abuse of notation we allow η_2 to stand for its image in any of the appropriate groups. Recall that a set $S \subset \Gamma$ is **$\boldsymbol{\Gamma}$-invariant** if $gsg^{-1} \in S$ for all $s \in S$ and $g \in \Gamma$. Note that the set $\{\mu^i\eta_2\mu^{-i} \mid i \in \mathbb{Z}\}$ is Γ-invariant where $\mu \in \Gamma$ generates $\pi/\pi^{(1)}$. Then we define the other S_n as follows:

Definition 7.1. Let $S_2 = S_2(\pi) \subset \mathbb{Q}[\pi^{(1)}/\pi^{(2)}_{\mathcal{S}}] \subset \mathbb{Q}[\pi/\pi^{(2)}_{\mathcal{S}}]$ be the multiplicative set generated by

$$\{\ q(a) \mid \widetilde{(q,p_2)} = 1,\ q(1) \neq 0,\ a \in A\} \cup \{\ p_2(\mu^i\eta_2\mu^{-i}) \mid i \in \mathbb{Z}\};$$

and for $2 < i \leq n$ let

$$S_n = S_n(\pi) = \{q_1(a_1)...q_r(a_r) \mid \widetilde{(p_n,q_j)} = 1;\ q_j(1) \neq 0;\ a_j \in \pi^{(n-1)}_{\mathcal{S}}/\pi^{(n)}_{\mathcal{S}}\}. \tag{7.5}$$

Since S_2 is a multiplicative subset of $\mathbb{Q}A$ that is Γ-invariant, it is a right divisor set of $\mathbb{Q}\Gamma$ by [8, Proposition 4.1]. Therefore Definition 3.6 applies to give a partially defined commutator series $\{\pi^{(i)}_{\mathcal{S}}\}$. Since $p_2(t) = q_{n-1}(t)$ is a knot polynomial, $p_2(t) \doteq p_2(t^{-1})$. Thus S_2 is closed (up to units) under the natural involution.

Lemma 7.2. *For each π and each $0 \leq i \leq n+1$*

$$\pi^{(i)}_{\mathcal{P}} \subset \pi^{(i)}_{\mathcal{S}}. \tag{7.6}$$

Proof. The proof is by induction on i. By Definition 3.6, $\pi^{(1)}_{\mathcal{P}} = \pi^{(1)}_r = \pi^{(1)}_{\mathcal{S}}$, so the Lemma is true for $i = 0, 1$. Suppose it is true for all values up to some fixed $i \geq 1$. Let $j : \pi \to \pi$ be the identity map. By [8, Proposition 3.2], it suffices to show that the induced ring map

$$j_* : \mathbb{Z}[\pi/\pi^{(i)}_{\mathcal{P}}] \to \mathbb{Z}[\pi/\pi^{(i)}_{\mathcal{S}}]$$

has the property that $j_*(S_{p_i}(\pi)) \subset S_i(\pi)$. For $i = 1$, j_* is the identity map and $S_1(\pi)$ is, by definition, identical to $S_{p_1}(\pi)$. It follows that $\pi^{(2)}_{\mathcal{P}} = \pi^{(2)}_{\mathcal{S}}$ as already observed in (7.4). Thus, for $i = 2$, j_* is again the identity map and, by Definitions 7.1 and 3.7, $S_2(\pi)$ strictly contains $S_{p_2}(\pi)$. For $i > 2$, the map j_*, although induced by the identity, will be a surjection with non-zero kernel. Nonetheless, by the inductive hypothesis, j induces a homomorphism

$$j_* : \pi^{(i-1)}_{\mathcal{P}}/\pi^{(i)}_{\mathcal{P}} \to \pi^{(i-1)}_{\mathcal{S}}/\pi^{(i)}_{\mathcal{S}}.$$

Recall from Definition 3.7 that

$$S_{p_i}(\pi) = \{q_1(a_1)...q_r(a_r) \mid \widetilde{(p_i,q_j)} = 1;\ q_j(1) \neq 0;\ a_j \in \pi^{(i-1)}_{\mathcal{P}}/\pi^{(i)}_{\mathcal{P}}\},$$

which is the multiplicative set generated by the described set of polynomials $q(a)$. If $q(a)$ is any such polynomial then $j_*(q(a)) = q(j_*(a))$ and since

$$a \in \frac{\pi^{(i-1)}_{\mathcal{P}}}{\pi^{(i)}_{\mathcal{P}}},\ j_*(a) \in \frac{\pi^{(i-1)}_{\mathcal{S}}}{\pi^{(i)}_{\mathcal{S}}}.$$

Thus, upon examining (7.5), we see that $q(j_*(a)) \in S_i(\pi)$. Hence $j_*(S_{p_i}(\pi)) \subset S_i(\pi)$ as desired. □

In particular this establishes (7.1).

Lemma 7.3. *The commutator series $\{\pi^{(i)}_{\mathcal{S}}\}$ is functorial with respect to any inclusion, $W_i \to W_j$, where $i > j$.*

Proof. Note that any such inclusion induces an isomorphism $H_1(W_i) \cong H_1(W_j) \cong \mathbb{Z} = \langle\mu\rangle$. If $\pi_{\mathcal{S}}^{(i)}$ were actually the polarized derived series localized at $\mathcal{P}$, then the functoriality would follow directly from our Theorem 3.10 [8, Thm. 4.16]. But since $\pi_{\mathcal{S}}^{(i)}$ is slightly different, we must actually repeat some of the proof of [8, Thm. 4.16]. Suppose $A = \pi_1(W_i), B = \pi_1(W_j)$ and $\psi : A \to B$ is induced by inclusion. We show, by induction on i, that $\psi(A_{\mathcal{S}}^{(i)}) \subset B_{\mathcal{S}}^{(i)}$. This holds for $i = 0$ so suppose it holds for $i = n$. We will show that $\psi(A_{\mathcal{S}}^{(n+1)}) \subset B_{\mathcal{S}}^{(n+1)}$. The induction hypothesis guarantees that, for each $1 \le k \le n$, ψ induces a homomorphism of pairs

$$\psi : (A/A_{\mathcal{S}}^{(k)}, A_{\mathcal{S}}^{(k-1)}/A_{\mathcal{S}}^{(k)}) \to (B/B_{\mathcal{S}}^{(k)}, B_{\mathcal{S}}^{(k-1)}/B_{\mathcal{S}}^{(k)}).$$

By [8, Prop.3.2] (or by examining (3.1)) it suffices to show that this map satisfies

$$\psi(S_k(A)) \subset S_k(B) \tag{7.7}$$

for each $1 \le k \le n$. First consider $k = 1$. Recall that

$$S_1(A) = \{q_1(\mu)...q_r(\mu) \mid (p_1(t), q_j(t)) = 1;\ A/A^{(1)} \cong \langle\mu\rangle\} \subset \mathbb{Q}[A/A^{(1)}].$$

Since ψ induces an isomorphism $\psi : A/A^{(1)} \to B/B^{(1)}$, $\psi(\mu) = \pm\mu$. By choosing generators once and for all, we may assume we that $\psi(\mu) = \mu$. So, for any such $q_j(t)$,

$$\psi(q_1(\mu)\dots q_r(\mu)) = q_1(\psi(\mu))\dots q_r(\psi(\mu)) = q_1(\mu)\dots q_r(\mu) \in S_1(B).$$

This verifies (7.7) for $k = 1$.

Now suppose $k > 1$. Recall that

$$S_k(A) = \{q_1(a_1)...q_r(a_r) \mid \widetilde{(p_n, q_j)} = 1;\ q_j(1) \neq 0;\ a_j \in A_{\mathcal{S}}^{(k-1)}/A_{\mathcal{S}}^{(k)}\}.$$

So, for any such $q_j(t)$,

$$\psi(q_1(a_1)\dots q_r(a_r)) = q_1(\psi(a_1))\dots q_r(\psi(a_r)) \in S_k(B),$$

since $\psi(a_j) \in B_{\mathcal{S}}^{(k-1)}/B_{\mathcal{S}}^{(k)}$.

Thus $\psi(S_k(A)) \subset S_k(B)$. ☐

7.1. **Establishing (7.2) and (7.3).**

Since $\pi_1(M_{K^0}) \subset \pi_1(W_0)$ is normally generated by its meridian, μ_0, and $\pi_1(M_{\overline{K}^0})$ is normally generated by its meridian, (that we denote) $\overline{\mu}_0$, the case $i = 0$ of the following Proposition will establish (7.2) and (7.3). Therefore the rest of the paper will be spent establishing Proposition 7.4.

Proposition 7.4. *For any i, $0 \le i \le n-2$, $\mu_i = \alpha_{i+1}$ is non-trivial, while $\overline{\mu}_i = \overline{\alpha}_{i+1}$ is trivial in*

$$\frac{\pi_1(W_i)^{(n-i)}}{\pi_1(W_i)_{\mathcal{S}}^{(n-i+1)}}.$$

To clarify the notation of this proposition, recall that, for $0 \le i \le n-1$, ∂W_i contains the disjoint union of the zero surgeries on the knots K^i (refer to the schematic Figure 7.2), and $\overline{K}^i$. Let μ_i and $\overline{\mu}_i$ denote the meridians of K^i and $\overline{K}^i$ in these copies of M_{K^i} and $M_{\overline{K}^i}$ respectively.

Also recall that $K^{i+1} = R^{i+1}_{\alpha_{i+1}}(K^i)$ for some circle α_{i+1} that generates the Alexander module of R^{i+1}; and $\overline{K}^{i+1} = \overline{R}^{i+1}_{\alpha_{i+1}}(\overline{K}^i)$. Let α_{i+1} denote (a push-off of) this circle in $M_{K^{i+1}} \subset \partial W_{i+1}$ (referring to Figure 7.2); and let $\overline{\alpha}_{i+1}$ denote (a push-off of) the other copy of α_{i+1} in $M_{\overline{K}^{i+1}} \subset \partial W_{i+1}$. Note that, by property (4) of Lemma 6.1, μ_i is isotopic to α_{i+1} in E_{i+1} and $\overline{\mu}_i$ is isotopic to $\overline{\alpha}_{i+1}$ in $\overline{E}_{i+1}$. Hence $\mu_i = \alpha_{i+1}$ and $\overline{\mu}_i = \overline{\alpha}_{i+1}$ as elements of $\pi_1(W_i)$.

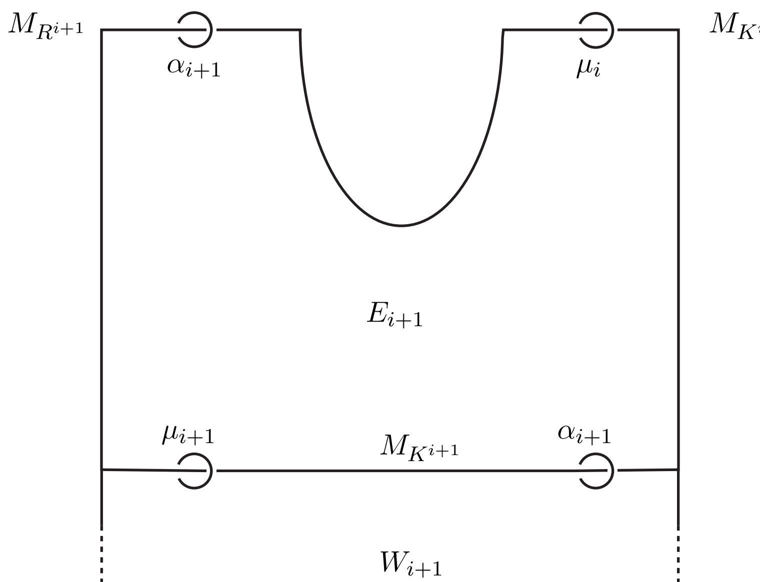

FIGURE 7.2. W_i

Proof of Proposition 7.4. The proof is by reverse induction on i, starting with $i = n-2$.

Before proving the base case $i = n-2$, we need to work out the "pre-base-case", $i = n-1$, where the situation is slightly different. Note that α_n and $\overline{\alpha}_n$ are what we have previously called η_1 and η_2 respectively.

Lemma 7.5. *$\mu_{n-1} = \eta_1$ and $\overline{\mu}_{n-1} = \eta_2$ are both non-trivial in*

$$\frac{\pi_1(W_{n-1})^{(1)}}{\pi_1(W_{n-1})^{(2)}_{\mathcal{S}}}.$$

Proof. Throughout the proof of this lemma we abbreviate $W = W_{n-1}$, $\pi = \pi_1(W_{n-1})$ and $\Delta = \Delta_n$. We make use of the fact that the integral and rational Alexander modules of a knot agree with those of its zero-framed surgery. Specifically we use $\mathcal{A}(K)$ to denote both the rational Alexander module of K and that of M_K. The inclusion maps induce a commutative diagram of maps between integral and rational Alexander modules as shown:

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{Z}}(\mathbb{E}^m) \arrow[r, "i"] \arrow[d, "i_1'"] & \mathcal{A}^{\mathbb{Z}}(\mathfrak{R}^m) \arrow[r, "j_*"] \arrow[dd, "i_2"] & \mathcal{A}^{\mathbb{Z}}(V) \arrow[r, "k_*"] \arrow[dd, "i_3"] & \mathcal{A}^{\mathbb{Z}}(W) \arrow[r] \arrow[dd, "i_4"] & \dfrac{\pi^{(1)}}{\pi^{(2)}_{\mathcal{S}}} \arrow[ddl, "i_5"] \\
\mathcal{A}(\mathbb{E}^m) \arrow[d, "i_1''"] & & & & \\
\mathcal{A}(\mathbb{E}^m)_{(\Delta)} \arrow[r, "i"] & \mathcal{A}(\mathfrak{R}^m)_{(\Delta)} \arrow[r, "j_*"] & \mathcal{A}(V)_{(\Delta)} \arrow[r, "k_*"] & \mathcal{A}(W)_{(\Delta)} &
\end{tikzcd}$$

Notice that $\mathcal{A}^{\mathbb{Z}}(\mathfrak{R}^m) \cong \mathcal{A}^{\mathbb{Z}}(K^n) \cong \mathcal{A}^{\mathbb{Z}}(M_{K^n}) \cong \mathcal{A}^{\mathbb{Z}}(\partial V)$. The maps j_* and k_* are induced by inclusion. The map i is induced by the connected sum decomposition, where here $\mathbb{E}^m$ denotes the "left-hand" copy in $\mathfrak{R}^m \equiv \mathbb{E}^m \# \mathbb{E}^m$. The existence and injectivity of i_5 is given by (7.4). Since the η_i represent elements in the Alexander module of $\mathbb{E}^m$, it suffices to show that the composition in the top row is injective. For this it suffices to show that the composition $k_* \circ j_* \circ i \circ i_1'' \circ i_1'$ is a monomorphism. Since it is well known that the integral Alexander modules $\mathcal{A}^{\mathbb{Z}}(\mathbb{E}^m) \cong \mathcal{A}^{\mathbb{Z}}(S^3 - \mathbb{E}^m)$ are $\mathbb{Z}$-torsion-free, i_1' is injective. Since $\mathcal{A}(\mathbb{E}^m)$ is a Δ-torsion module, i_1'' is injective. Under the connected sum decomposition the localized Alexander module of $\mathfrak{R}^m$ decomposes as the direct sum of the localized Alexander modules of its summands $\mathbb{E}^m$. The Blanchfield form decomposes similarly. Hence i is injective. Now consider the map j_* induced by the inclusion $\partial V \hookrightarrow V$.

$$j_* : \mathcal{A}(\partial V)_{(\Delta)} \cong \mathcal{A}(\mathfrak{R}^m)_{(\Delta)} \equiv H_1(M_{\mathfrak{R}^m}; \mathbb{Q}[t,t^{-1}]S_{p_1}^{-1}) \to H_1(V; \mathbb{Q}[t,t^{-1}]S_{p_1}^{-1}) \equiv \mathcal{A}(V)_{(\Delta)}.$$

Since V is an $(n.5, \mathcal{P})$-solution for ∂V, and $\pi^{(i)}_{\mathcal{P}} \subset \pi^{(i)}_{\mathcal{S}}$, V is an $(n.5, \mathcal{S})$ solution, so it is certainly a $(1, \mathcal{S})$-solution. Consider the coefficient system $\psi : \pi_1(V) \to \pi_1(V)/\pi_1(V)^{(1)}_{\mathcal{S}} \cong \mathbb{Z}$ (recall $G^{(1)}_{S} = G^{(1)}_{r}$ for any group G). Then [10, Theorem 7.15] applies to say that the kernel of j_* is isotropic with respect to the classical Blanchfield form on $\mathcal{A}(\mathfrak{R}^m)_{(\Delta)}$. Hence the kernel, P, of $i \circ j_*$ is isotropic with respect to the classical Blanchfield form on $\mathcal{A}(\mathbb{E}^m)_{(\Delta)}$. But, since the Alexander polynomial of $\mathbb{E}^m$ is irreducible by Proposition 2.3, the rational Alexander module of $\mathbb{E}^m$ has no proper submodules. The case $P = \mathcal{A}(\mathbb{E}^m)_{(\Delta)}$ is not possible since the localized classical Blanchfield form is non-singular and $\mathcal{A}(\mathbb{E}^m)_{(\Delta)} \neq 0$. Thus $P = 0$ so $i \circ j_*$ is injective.

It only remains to show that the lower map k_* is injective (actually an isomorphism). Since localization is an exact functor, this is equivalent to showing that the inclusion map induces an isomorphism between the rational Alexander modules of V and W. Recall that $W = W_n = V \cup E_n$. Recall from property (1) of Lemma 6.1 applied to E_n, that the kernel on π_1 of the inclusion $M_{K^n} = \partial V \to E_n$ is normally generated by the longitudes of the infecting knots K^{n-1} and $\overline{K}^{n-1}$ as curves in $\pi_1(M_{K^n})$. These lie in the second derived subgroups of $\pi_1(S^3 - K^{n-1})$ and $\pi_1(S^3 - \overline{K}^{n-1})$ respectively and so lie in the third derived subgroup of $\pi_1(M_{K^n})$ (refer to

Figure 2.8). Since the rational Alexander module of any space X with $H_1(X) \cong \mathbb{Z}$ may be described as $G^{(1)}/G^{(2)} \otimes \mathbb{Q}$ where $G = \pi_1(X)$, this shows that the rational Alexander modules of V and W are isomorphic. $\square$

The crucial base case, $i = n-2$, in the (reverse) inductive proof of Proposition 7.4 is:

Lemma 7.6. *$\mu_{n-2} = \alpha_{n-1}$ is non-trivial, while $\overline{\mu}_{n-2} = \overline{\alpha}_{n-1}$ is trivial in*

$$\frac{\pi_1(W_{n-2})^{(2)}}{\pi_1(W_{n-2})^{(3)}_{\mathcal{S}}}.$$

Proof. It might be helpful to refer to Figure 7.2 with $i = n-2$. By property (1) of Lemma 6.1, the kernel of the map

$$\pi_1(W_{n-1}) \to \pi_1(W_{n-1} \cup E_{n-1} \cup \overline{E}_{n-1}) = \pi_1(W_{n-2})$$

is normally generated by the longitudes, $\ell_{n-2}, \overline{\ell}_{n-2}$, of the infecting knots K^{n-2} and $\overline{K}^{n-2}$ viewed as curves in $S^3 \setminus K^{n-2} \subset M_{K^{n-1}} \subset \partial W_{n-1}$ and $S^3 \setminus \overline{K}^{n-2} \subset M_{\overline{K}^{n-1}} \subset \partial W_{n-1}$. But of course these lie in the second derived subgroups of $\pi_1(S^3 \setminus K^{n-2})$ and $\pi_1(S^3 \setminus \overline{K}^{n-2})$ respectively, and so lie in the second derived subgroups of $\pi_1(M_{K^{n-1}})$ and $\pi_1(M_{\overline{K}^{n-1}})$ respectively. But, as observed in Lemma 7.5

$$\pi_1(M_{K^{n-1}}) = \langle \mu_{n-1} \rangle \subset \pi_1(W_{n-1})^{(1)}, \tag{7.8}$$

and similarly for $\pi_1(M_{\overline{K}^{n-1}})$. It follows that both ℓ_{n-2} and $\overline{\ell}_{n-2}$ lie the third derived subgroup of $\pi_1(W_{n-1})$ and hence lie in $\pi_1(W_{n-1})^{(3)}_{\mathcal{S}}$. Thus the inclusion $W_{n-1} \to W_{n-2}$ induces an isomorphism

$$\frac{\pi_1(W_{n-1})}{\pi_1(W_{n-1})^{(3)}_{\mathcal{S}}} \cong \frac{\pi_1(W_{n-2})}{\pi_1(W_{n-2})^{(3)}_{\mathcal{S}}},$$

by weak functoriality and by [8, Prop.4.7].

Therefore, to prove Lemma 7.6, it suffices to let $\pi = \pi_1(W_{n-1})$, and show that α_{n-1} is non-trivial in $\pi^{(2)}/\pi^{(3)}_{\mathcal{S}}$ and that $\overline{\alpha}_{n-1}$ is trivial in $\pi^{(2)}/\pi^{(3)}_{\mathcal{S}}$. Throughout the rest of the proof of Lemma 7.6, we will abbreviate $W = W_{n-1}$, $\pi = \pi_1(W_{n-1})$, $J = K^{n-1}$ and $\overline{J} = \overline{K}^{n-1}$. Thus $\partial W = M_{\mathfrak{R}^m} \cup M_J \cup M_{\overline{J}}$.

Consider the following commutative diagram (which we justify below) where $\Gamma = \pi/\pi^{(2)}_{\mathcal{S}}$ and $\mathcal{R} = \mathbb{Q}\Gamma S_2^{-1}$. Since we may view $\alpha_{n-1} \in \pi_1(M_J)^{(1)}$ and $\overline{\alpha}_{n-1} \in \pi_1(M_{\overline{J}})^{(1)}$, we have reduced Lemma 7.6 to showing that α_{n-1} is *not* in the kernel of the top row of the diagram while $\overline{\alpha}_{n-1}$

does lie in this kernel.

$$\begin{tikzcd}
\pi_1(M_J)^{(1)} \oplus \pi_1(M_{\overline{J}})^{(1)} \arrow[rr, "j_*"] \arrow[d, "\pi"] & & \pi^{(2)} \arrow[r, "\phi"] \arrow[d] & \frac{\pi_{\mathcal{S}}^{(2)}}{\pi_{\mathcal{S}}^{(3)}} \arrow[d, "j"] \\
(\mathcal{A}(J) \oplus \mathcal{A}(\overline{J})) \otimes \mathcal{R} \arrow[r, "\cong"] & H_1(M_J \cup M_{\overline{J}}; \mathcal{R}) \arrow[r, "j_*"] & H_1(W; \mathcal{R}) \arrow[r, "\cong"] & \frac{\pi_{\mathcal{S}}^{(2)}}{[\pi_{\mathcal{S}}^{(2)}, \pi_{\mathcal{S}}^{(2)}]} \otimes \mathcal{R}
\end{tikzcd}$$

The j_* in the upper row of the diagram is justified by our observation (7.8), which says that $\pi_1(M_J) \subset \pi^{(1)}$ and $\pi_1(M_{\overline{J}}) \subset \pi^{(1)}$. Now we consider the first map in the bottom row. By Lemma 7.5 the coefficient system $\pi \to \Gamma$, when restricted to $\pi_1(M_J)$ is non-trivial:

$$\pi_1(M_J) = \langle \mu_{n-1} \rangle \hookrightarrow \frac{\pi^{(1)}}{\pi_{\mathcal{S}}^{(2)}} \hookrightarrow \frac{\pi}{\pi_{\mathcal{S}}^{(2)}} \equiv \Gamma,$$

but also factors through $\pi_1(M_J)/\pi_1(M_J)^{(1)} \cong \mathbb{Z}$ using (7.8). It follows that

$$H_1(M_J; \mathbb{Q}\Gamma) \cong H_1(M_J; \mathbb{Q}[t, t^{-1}]) \otimes \mathbb{Q}\Gamma \equiv \mathcal{A}(J) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Gamma,$$

where $\mathbb{Q}[t, t^{-1}]$ acts on $\mathbb{Q}\Gamma$ by $t \to \mu_{n-1}$ (equivalently $t \to \eta_1$). Hence

$$H_1(M_J; \mathcal{R}) \cong \mathcal{A}(J) \otimes \mathcal{R};$$

and similarly for $\overline{J}$, where t acts by $\overline{\mu}_{n-1} = \eta_2$. This explains the first map in the lower row of the diagram. To justify the last map in the lower row, recall that $H_1(W; \mathbb{Z}\Gamma)$ has an interpretation as the first homology module of the Γ-covering space of W. The fundamental group of this covering space is the kernel of $\pi \to \Gamma$. Hence

$$H_1(W; \mathbb{Z}\Gamma) \cong \frac{\pi_{\mathcal{S}}^{(2)}}{[\pi_{\mathcal{S}}^{(2)}, \pi_{\mathcal{S}}^{(2)}]}$$

Since the Ore localization $\mathcal{R}$ is a flat $\mathbb{Z}\Gamma$-module, the $\cong$ is justified. This completes the explanation of the diagram. Since, by Definitions 3.7 and 7.1,

$$\pi_{\mathcal{S}}^{(3)} = \ker \left(\pi_{\mathcal{S}}^{(2)} \to \frac{\pi_{\mathcal{S}}^{(2)}}{[\pi_{\mathcal{S}}^{(2)}, \pi_{\mathcal{S}}^{(2)}]} \to \frac{\pi_{\mathcal{S}}^{(2)}}{[\pi_{\mathcal{S}}^{(2)}, \pi_{\mathcal{S}}^{(2)}]} \otimes \mathbb{Q}\Gamma S_2^{-1} \right),$$

it follows that the vertical map j (in the diagram) is injective. Hence, to establish Lemma 7.6, it suffices to show that the class represented by $\alpha_{n-1} \otimes 1$ is *not* in the kernel of the bottom row of the diagram while that represented by $\overline{\alpha}_{n-1} \otimes 1$ *does* lie in this kernel.

Recall that $\overline{J} \equiv \overline{K}^{n-1} \equiv \overline{R}^{n-1}_{\overline{\alpha}_{n-1}}(\overline{K}^{n-2})$ where $\overline{\alpha}_{n-1}$ generates $\mathcal{A}(\overline{R}^{n-1})$ (note this implies the latter module is cyclic). Therefore $\mathcal{A}(\overline{J}) \cong \mathcal{A}(\overline{R}^{n-1})$. By hypothesis, the Alexander polynomial of R^{n-1} is $q_{n-1}(t) = p_2(t)$. Thus

$$\langle \overline{\alpha}_{n-1} \rangle \cong \mathcal{A}(\overline{J}) \cong \frac{\mathbb{Q}[t, t^{-1}]}{p_2(t)\mathbb{Q}[t, t^{-1}]}$$

and

$$\langle\overline{\alpha}_{n-1}\otimes 1\rangle \cong \mathcal{A}(\overline{J})\otimes\mathcal{R}\cong\left(\frac{\mathbb{Q}\Gamma}{p_2(\eta_2)\mathbb{Q}\Gamma}\right)S_2^{-1}\cong 0,$$

where the last equality holds since $p_2(\eta_2)\in S_2$, by Definition 7.1 (see [8, Thm. 4.12] for more detail). Therefore $\overline{\alpha}_{n-1}\otimes 1$ lies in the kernel of the bottom row of the diagram.

Suppose that $\alpha_{n-1}\otimes 1$ *were* in the kernel of the bottom row of the diagram. We shall reach a contradiction. Recall that $W_{n-1}\equiv V\cup E_n$. Recall that V is an $(n.5,\mathcal{P})$-solution. Since $n\geq 2$, V is a $(2,\mathcal{P})$-solution. One easily checks that

$$\frac{H_2(W_{n-1})}{i_*(H_2(\partial W_{n-1})}\cong H_2(V).$$

Hence this group has a basis consisting of surfaces that satisfy parts (2) and (3) of Definition 3.3 (with $n=2$). But W_{n-1} fails to satisfy part (1) of that definition and ∂W_{n-1} is disconnected. Such a manifold was named a $(2,\mathcal{P})$**-bordism** in [8, Definition 7.11]. By [8, Thms. 7.14, 7.15], if P is the kernel of the map

$$j^*: H_1(M_J;\mathcal{R})\to H_1(W;\mathcal{R}),$$

as in the bottom row of the diagram, then P is an isotropic submodule for the Blanchfield linking form on $H_1(M_J;\mathcal{R})$. Since we have supposed that $\alpha_{n-1}\otimes 1\in P$ and since this element is a generator of $H_1(M_J;\mathcal{R})$, it would follow that this Blanchfield form were identically zero on $H_1(M_J;\mathcal{R})$. But by [8, Lemma 7.16] this form is non-singular. This would imply that $H_1(M_J;\mathcal{R})$ were the zero module. This is a contradiction once we show that

$$\mathcal{A}(J)\otimes\mathcal{R}\cong\left(\frac{\mathbb{Q}\Gamma}{p_2(\eta_1)\mathbb{Q}\Gamma}\right)S_2^{-1}\neq 0. \tag{7.9}$$

This is a non-trivial result since we are dealing with a noncommutative localization.

Note that, by the hypotheses of Theorem 5.4, $p_2(t)=q_{n-1}(t)$ is not a unit in $\mathbb{Q}[t,t^{-1}]$. The map $\mathbb{Z}\to\Gamma$ given by $t\to\eta_1$ is not zero by Lemma 7.5. Since Γ is PTFA, it is torsion-free, so $\langle\eta_1\rangle\subset\Gamma$. Hence $\mathbb{Q}\Gamma$ is a free left $\mathbb{Q}[\eta_1,\eta_1^{-1}]$-module on the right cosets of $\langle\eta_1\rangle\in\Gamma$ [43, Chapter 1, Lemma 1.3]. Thus, upon fixing a set of coset representatives, any $x\in\mathbb{Q}\Gamma$ has a unique decomposition

$$x=\Sigma_\gamma x_\gamma\gamma,$$

where $x_\gamma\in\mathbb{Q}[\eta_1,\eta_1^{-1}]$ and the sum is over a set of coset representatives $\{\gamma\in\Gamma\}$. It follows that $p_2(\eta_1)$ has no right inverse in $\mathbb{Q}\Gamma$ since if $p_2(\eta_1)x=1$ then

$$p_2(\eta_1)x=p_2(\eta_1)\Sigma_\gamma x_\gamma\gamma=\Sigma_\gamma p_2(\eta_1)x_\gamma\gamma=1.$$

Looking at the coset $\gamma=e$, we have $p_2(\eta_1)x_e=1$ in $\mathbb{Q}[\eta_1,\eta_1^{-1}]$, contradicting the fact that $p_2(t)$ is not a unit in $\mathbb{Q}[t,t^{-1}]$. Therefore, since $\mathbb{Q}\Gamma$ is a domain,

$$\frac{\mathbb{Q}\Gamma}{p_2(\eta_1)\mathbb{Q}\Gamma}\ncong 0.$$

Continuing, by [46, Corollary 3.3, p. 57], the kernel of

$$\frac{\mathbb{Q}\Gamma}{p_2(\eta_1)\mathbb{Q}\Gamma} \to \left(\frac{\mathbb{Q}\Gamma}{p_2(\eta_1)\mathbb{Q}\Gamma}\right) S_2^{-1}$$

is precisely the S_2-torsion submodule. Hence to establish (7.9), it suffices to show that the generator of $\mathbb{Q}\Gamma/p_2(\eta_1)\mathbb{Q}\Gamma$ is not S_2-torsion. Suppose [1] were S_2-torsion. We will show that $[1]=0$, implying that $\mathbb{Q}\Gamma/p_2(\eta_1)\mathbb{Q}\Gamma$ is S_2-torsion-free. If [1] were S_2-torsion then $1s = p_2(\eta_1)y$ for some $s \in S_2$ and for some $y \in \mathbb{Q}\Gamma$. We examine this equation in $\mathbb{Q}\Gamma$.

Recall that $\Gamma = \pi/\pi^{(2)}_{\mathcal{S}}$. Let $A = \pi^{(1)}/\pi^{(2)}_{\mathcal{S}} \lhd \Gamma$. Since $A \subset \Gamma$, $\mathbb{Q}\Gamma$, viewed as a left $\mathbb{Q}A$-module, is free on the right cosets of A in Γ. Thus any $y \in \mathbb{Q}\Gamma$ has a unique decomposition

$$y = \Sigma_\gamma y_\gamma \gamma,$$

where sum is over a set of coset representatives $\{\gamma \in \Gamma\}$ and $y_\gamma \in \mathbb{Q}A$. Therefore we have

$$s = p_2(\eta_1)\Sigma_\gamma y_\gamma \gamma.$$

Recall from Definition 7.1 that $s \in S_2 \subset \mathbb{Q}A$. It follows that for each coset representative $\gamma \neq e$ we have $0 = p_2(\eta_1)y_\gamma$ so $y_\gamma = 0$ (note that $p_2(\eta_1) \neq 0$ since $\mathbb{Q}[\eta_1^{\pm 1}] \subset \mathbb{Q}\Gamma$). Hence $y \in \mathbb{Q}A$ and we have

$$s = p_2(\eta_1)y \tag{7.10}$$

as an equation in $\mathbb{Q}A$. Recall from Definition 7.1 that an arbitrary element of S_2 is a product of terms of the form $q(a)$ and terms of the form $p_2(\mu^i\eta_2\mu^{-i})$ for some $a \in A$, $q(t)$ in $\mathbb{Q}[t,t^{-1}]$ where $\widetilde{(p_2,q)} = 1$, $q(1) \neq 0$, and μ generates $\pi/\pi^{(1)}$. Since A is a torsion-free abelian group, (7.10) may be viewed as an equation in $\mathbb{Q}F$ for some free abelian group $F \subset A$ of finite rank r. Since $\mathbb{Q}F$ is a UFD and since $\widetilde{(p_2,q)} = 1$ we can apply the following.

Proposition 7.7 ([8, Prop.4.5])**.** *Suppose $p(t), q(t) \in \mathbb{Q}[t,t^{-1}]$ are non-zero. Then p and q are strongly coprime if and only if, for any finitely-generated free abelian group F and any nontrivial $a, b \in F$, $p(a)$ is relatively prime to $q(b)$ in $\mathbb{Q}F$ (a unique factorization domain).*

Thus the greatest common divisor, in $\mathbb{Q}F$, of $p_2(\eta_1)$ and $q(a)$ is a unit (note that if a is trivial in F then $q(a) = q(1) \neq 0$ is itself a unit). Thus $p_2(\eta_1)$ divides the product of the terms of the form $p_2(\mu^i\eta_2\mu^{-i})$. Choose a basis, $\{x, x_2, \dots, x_r\}$, for F in which $\eta_1 = x^r$ for some $r > 0$ (since $\eta_1 \neq 0$ by Lemma 7.5) and $\mu^i\eta_2\mu^{-i} = x^{n_i}x_2^{n_{i,2}}...x_r^{n_{i,r}}$. Then we may regard $\mathbb{Q}F$ as a Laurent polynomial ring in the variables $\{x, x_2, \dots, x_r\}$. Since p_2 is not zero and not a unit, there exists a non-zero complex root $x = \tau$ of $p_2(x^r)$. Suppose $\tilde{p}(x)$ is an irreducible factor (in $\mathbb{Q}F$) of $p_2(x^r)$ of which τ is a root. Then, for some i, $\tilde{p}(x)$ divides $p_2(x^{n_i}x_2^{n_{i,2}}...x_r^{n_{i,r}})$. Then τ must be a zero of $p_2(x^{n_i}x_2^{n_{i,2}}...x_r^{n_{i,r}})$ for *every* complex value of $x_2, ..., x_r$. This is impossible unless each $n_{i,j} = 0$. Thus, for this value of i, $\mu^i\eta_2\mu^{-i} = x^n$, in F, for some n. Note $n \neq 0$ since η_2 is nontrivial by Lemma 7.5. Thus

$$\mu^i\eta_2^r\mu^{-i} = (\mu^i\eta_2\mu^{-i})^r = x^{nr} = \eta_1^n, \tag{7.11}$$

for some i and some non-zero integers n and r. This equation holds in A. However, the circles μ, η_2 and η_1 all live in $M_{\mathfrak{R}^m}$ and in fact can be interpreted in $\mathcal{A}^{\mathbb{Z}}(\mathbb{E}^m)$ (the left-hand copy of

$\mathbb{E}^m$). But recall that in the proof of Lemma 7.5 we showed that the map

$$\mathcal{A}^{\mathbb{Z}}(\mathbb{E}^m) \to \mathcal{A}^{\mathbb{Z}}(W) \to \frac{\pi^{(1)}}{\pi^{(2)}_{\mathcal{S}}} \equiv A$$

is injective. Hence if (7.11) holds in A then it holds as an equation in $\mathcal{A}^{\mathbb{Z}}(\mathbb{E}^m)$, and hence also in $\mathcal{A}(\mathbb{E}^m)$, where, in module notation, it has the form

$$(t_*)^i(r\eta_2) = n\eta_1.$$

But the simple computation in the following Lemma proves that this is impossible.

Lemma 7.8. *Let m be a non-zero integer, let $\mathbb{E}^m$ be the knot of Figure 2.5 and let $\langle\eta_i\rangle$, $i = 1, 2$ be the subspace of $\mathcal{A}(\mathbb{E}^m)$ generated by the circle η_i shown in Figure 2.9. Then, under the automorphism*

$$t_* : \mathcal{A}(\mathbb{E}^m) \to \mathcal{A}(\mathbb{E}^m),$$

for every integer k, $(t_*)^k(\langle\eta_2\rangle) \cap \langle\eta_1\rangle = \vec{0}$.

Proof. We may assume that $m > 0$. If V is the Seifert matrix for $\mathbb{E}^m$ as in the proof of Proposition 2.3, with respect to the basis $\{a_i\}$ consisting of the cores of the obvious bands where $\ell k(a_i, \eta_i) = 1$, then the rational Alexander module is presented by $V - tV^T$ with respect to the basis $\{\eta_1, \eta_2\}$ where the relations are given by the columns, that is, $(V - tV^T)\vec{v} = \vec{0}$ for all $\vec{v}$. Since V has non-zero determinant, upon left multiplying the latter equation by V^{-1}, one recovers the fact that the automorphism t^* is given by left multiplication by $(V^{-1})^T V$. Hence

$$t^* = \frac{1}{m^2}\begin{pmatrix} m^2+1 & m \\ m & m^2 \end{pmatrix} = \frac{1}{m^2}M,$$

for M as indicated, with respect to the basis $\{\eta_1, \eta_2\}$. It then suffices to prove that, for any k, there is no non-zero solution (x_0, y_0) to the equation

$$M^k \begin{pmatrix} 0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}.$$

If there were such a solution (x_0, y_0) then there would be one with $y_0 > 0$. Let $B = \{(x, y) \mid x \geq 0,\ y > 0\}$. Since

$$\begin{pmatrix} m^2+1 & m \\ m & m^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (m^2+1)x + my \\ mx + m^2 y \end{pmatrix}$$

we observe that $M(B) \subset B$. But then if $k \geq 0$, $M^k(B) \subset B$. This is a contradiction since $(0, y_0) \in B$ but $(x_0, 0) \notin B$. Therefore there is no non-zero solution if $k \geq 0$. If $k < 0$ then we have

$$\begin{pmatrix} 0 \\ y_0 \end{pmatrix} = M^{-k}\begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

where $-k = s > 0$. As above if there were a non-zero solution then there would be one with $x_0 > 0$. Letting $A = \{(x, y) \mid x > 0,\ y \geq 0\}$, we observe that $M^s(A) \subset A$, $(x_0, 0) \in A$ and $(0, y_0) \notin A$, which is a contradiction. □

This contradiction establishes (7.9), finally finishing the proof of Lemma 7.6. □

We now complete the induction step in the proof of Proposition 7.4.

Suppose, for some i, $1 \leq i \leq n-2$, Proposition 7.4 holds, that is, $\mu_i = \alpha_{i+1}$ is non-trivial, while $\overline{\mu}_i = \overline{\alpha}_{i+1}$ is trivial in

$$\frac{\pi_1(W_i)^{(n-i)}}{\pi_1(W_i)^{(n-i+1)}_{\mathcal{S}}}. \tag{7.12}$$

To complete the inductive step we need to show that

$$\overline{\mu}_{i-1} = \overline{\alpha}_i = 0 \ \in \frac{\pi_1(W_{i-1})^{(n-i+1)}}{\pi_1(W_{i-1})^{(n-i+2)}_{\mathcal{S}}}. \tag{7.13}$$

and show that

$$\mu_{i-1} = \alpha_i \neq 0 \ \in \frac{\pi_1(W_{i-1})^{(n-i+1)}}{\pi_1(W_{i-1})^{(n-i+2)}_{\mathcal{S}}}. \tag{7.14}$$

By the inductive hypothesis and weak functoriality,

$$\overline{\mu}_i \in \pi_1(W_i)^{(n-i+1)}_{\mathcal{S}} \subset \pi_1(W_{i-1})^{(n-i+1)}_{\mathcal{S}}.$$

But, by property (1) of Lemma 6.1, $\overline{\mu}_i \in \pi_1(M_{\overline{K}^i})$ normally generates $\pi_1(\overline{E}_i)$ so

$$\pi_1(\overline{E}_i) \subset \pi_1(W_{i-1})^{(n-i+1)}_{\mathcal{S}},$$

and so by property (1) of Proposition 3.2,

$$[\pi_1(\overline{E}_i), \pi_1(\overline{E}_i)] \subset [\pi_1(W_{i-1})^{(n-i+1)}_{\mathcal{S}}, \pi_1(W_{i-1})^{(n-i+1)}_{\mathcal{S}}] \subset \pi_1(W_{i-1})^{(n-i+2)}_{\mathcal{S}}.$$

Since $\ell k(\overline{\alpha}_i, \overline{R}^i) = 0$,

$$\overline{\alpha}_i \in [\pi_1(M_{\overline{K}^i}), \pi_1(M_{\overline{K}^i})] \subset [\pi_1(\overline{E}_i), \pi_1(\overline{E}_i)] \subset \pi_1(W_{i-1})^{(n-i+2)}_{\mathcal{S}}.$$

This proves (7.13).

Now to we need to prove (7.14). By property (1) of Lemma 6.1, the kernel of the map

$$\pi_1(W_i) \to \pi_1(W_i \cup E_i \cup \overline{E}_i) = \pi_1(W_{i-1})$$

is normally generated by the longitudes, $\ell_{i-1}, \overline{\ell}_{i-1}$, of the infecting knots K^{i-1} and $\overline{K}^{i-1}$ viewed as curves in $S^3 \setminus K^{i-1} \subset M_{K^i} \subset \partial W_i$ and $S^3 \setminus \overline{K}^{i-1} \subset M_{\overline{K}^i} \subset \partial W_i$. But of course these lie in the second derived subgroups of $\pi_1(S^3 \setminus K^{i-1})$ and $\pi_1(S^3 \setminus \overline{K}^{i-1})$ respectively, and so lie in the second derived subgroups of $\pi_1(M_{K^i})$ and $\pi_1(M_{\overline{K}^i})$ respectively. But, by the induction hypothesis (7.12),

$$\pi_1(M_{K^i}) = \langle \mu_i \rangle \subset \pi_1(W_i)^{(n-i)}, \tag{7.15}$$

and similarly for $\pi_1(M_{\overline{K}^i})$. It follows that both ℓ_{i-1} and $\overline{\ell}_{i-1}$ lie in

$$\pi_1(W_i)^{(n-i+2)} \subset \pi_1(W_i)^{(n-i+2)}_{\mathcal{S}}.$$

Thus the inclusion $W_i \to W_{i-1}$ induces an isomorphism

$$\frac{\pi_1(W_i)^{(n-i+1)}}{\pi_1(W_i)_{\mathcal{S}}^{(n-i+2)}} \cong \frac{\pi_1(W_{i-1})^{(n-i+1)}}{\pi_1(W_{i-1})_{\mathcal{S}}^{(n-i+2)}},$$

by weak functoriality and by [8, Prop. 4.7].

Consequently, to establish (7.14), it suffices to let $\pi = \pi_1(W_i)$, and show that α_i is non-trivial in $\pi^{(n-i+1)}/\pi_{\mathcal{S}}^{(n-i+2)}$. Throughout the rest of the proof, we will abbreviate $W = W_i$, $\pi = \pi_1(W_i)$, $J = K^i$ and $\overline{J} = \overline{K}^i$. Thus $M_J \subset \partial W$.

Consider the following commutative diagram (which we justify below) where $\Gamma = \pi/\pi_{\mathcal{S}}^{(n-i+1)}$ and $\mathcal{R} = \mathbb{Q}\Gamma S_{n-i+1}^{-1}$. Since $\alpha_i \in \pi_1(M_J)^{(1)}$ we have reduced (7.14) to showing that α_i is *not* in the kernel of the top row of the diagram.

$$\begin{tikzcd}
\pi_1(M_J)^{(1)} \arrow[rr, "j_*"] \arrow[d, "\pi"] & & \pi^{(n-i+1)} \arrow[r, "\phi"] \arrow[d] & \dfrac{\pi_{\mathcal{S}}^{(n-i+1)}}{\pi_{\mathcal{S}}^{(n-i+2)}} \arrow[d, "j"] \\
\mathcal{A}(J) \otimes \mathcal{R} \arrow[r, "\cong"] & H_1(M_J;\mathcal{R}) \arrow[r, "j_*"] & H_1(W;\mathcal{R}) \arrow[r, "\cong"] & \dfrac{\pi_{\mathcal{S}}^{(n-i+1)}}{[\pi_{\mathcal{S}}^{(n-i+1)}, \pi_{\mathcal{S}}^{(n-i+1)}]} \otimes \mathcal{R}
\end{tikzcd}$$

The j_* in the upper row of the diagram is justified by (7.15). Now we consider the first map in the bottom row. By the inductive hypothesis (7.14) the coefficient system $\pi \to \Gamma$, when restricted to $\pi_1(M_J)$ is non-trivial:

$$\pi_1(M_J) = \langle \mu_i \rangle \hookrightarrow \frac{\pi^{(n-i)}}{\pi_{\mathcal{S}}^{(n-i+1)}} \hookrightarrow \frac{\pi}{\pi_{\mathcal{S}}^{(n-i+1)}} \equiv \Gamma,$$

but also factors through $\pi_1(M_J)/\pi_1(M_J)^{(1)} \cong \mathbb{Z}$ because of (7.15). It follows that

$$H_1(M_J;\mathbb{Q}\Gamma) \cong H_1(M_J;\mathbb{Q}[t,t^{-1}]) \otimes \mathbb{Q}\Gamma \equiv \mathcal{A}(J) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Gamma,$$

where $\mathbb{Q}[t,t^{-1}]$ acts on $\mathbb{Q}\Gamma$ by $t \to \mu_i$. Hence

$$H_1(M_J;\mathcal{R}) \cong \mathcal{A}(J) \otimes \mathcal{R}.$$

To justify the last map in the lower row, recall that $H_1(W;\mathbb{Z}\Gamma)$ has an interpretation as the first homology module of the Γ-covering space of W corresponding to the kernel of $\pi \to \Gamma$. Hence

$$H_1(W;\mathbb{Z}\Gamma) \cong \frac{\pi_{\mathcal{S}}^{(n-i+1)}}{[\pi_{\mathcal{S}}^{(n-i+1)}, \pi_{\mathcal{S}}^{(n-i+1)}]}$$

This completes the explanation of the diagram. Since, by Definitions 3.7 and 7.1,

$$\pi_{\mathcal{S}}^{(n-i+2)} = \ker\left(\pi_{\mathcal{S}}^{(n-i+1)} \to \frac{\pi_{\mathcal{S}}^{(n-i+1)}}{[\pi_{\mathcal{S}}^{(n-i+1)}, \pi_{\mathcal{S}}^{(n-i+1)}]} \to \frac{\pi_{\mathcal{S}}^{(n-i+1)}}{[\pi_{\mathcal{S}}^{(n-i+1)}, \pi_{\mathcal{S}}^{(n-i+1)}]} \otimes \mathbb{Q}\Gamma S_{n-i+1}^{-1}\right),$$

it follows that the vertical map j (in the diagram) is injective. Hence, to establish (7.14), it suffices to show that the class represented by $\alpha_i \otimes 1$ is *not* in the kernel of the bottom row of the diagram.

Recall that $J \equiv K^i \equiv R^i_{\alpha_i}(K^{i-1})$ where α_i generates $\mathcal{A}(R^i)$. Therefore $\mathcal{A}(J) \cong \mathcal{A}(R^i)$. By the hypotheses of Theorem 5.4, the Alexander polynomial of R^i is $q_i(t) = p_{n-i+1}(t)$. Thus

$$\langle \alpha_i \otimes 1 \rangle \cong \mathcal{A}(J) \otimes \mathcal{R} \cong \left(\frac{\mathbb{Q}\Gamma}{p_{n-i+1}(\mu_i)\mathbb{Q}\Gamma} \right) S^{-1}_{p_{n-i+1}}. \tag{7.16}$$

where the last equality holds because, since $1 \leq i \leq n-2$, it follows that $3 \leq n-i+1 \leq n$, so $S_{n-i+1} = S_{p_{n-i+1}}$, by Definition 7.1.

Suppose that $\alpha_i \otimes 1$ *were* in the kernel of the bottom row of the diagram. We shall reach a contradiction. Recall that

$$W = W_i \equiv V \cup E_n \cup E_{n-1} \cup \overline{E}_{n-1} \cup \cdots \cup E_{i+1} \cup \overline{E}_{i+1}.$$

Recall also that V is an $(n.5, \mathcal{P})$-solution. Thus, by (7.6), V is an $(n.5, \mathcal{S})$-solution and, since $n-i+1 \leq n$, V is also an $(n-i+1, \mathcal{S})$-solution. One easily checks that

$$\frac{H_2(W_i)}{i_*(H_2(\partial W_i))} \cong H_2(V).$$

Hence this group has a basis consisting of surfaces that satisfy parts (2) and (3) of Definition 3.3 (with $n-i+1$). Thus W_i is an $(n-i+1, \mathcal{S})$-bordism ([8, Definition 7.11]). By [8, Thms. 7.14, 7.15], if P is the kernel of the map

$$j_* : H_1(M_J; \mathcal{R}) \to H_1(W; \mathcal{R}),$$

then P is isotropic for the Blanchfield linking form on $H_1(M_J; \mathcal{R})$. Therefore if the generator $\alpha_i \otimes 1$ were in P, it would follow that this Blanchfield form were identically zero on $H_1(M_J; \mathcal{R})$. But by [8, Lemma 7.16] this form is non-singular. This would imply that $H_1(M_J; \mathcal{R}) = 0$. This is a contradiction once we show that (7.16) is in fact a non-trivial module. It is shown in [8, Theorem 4.12] that

$$\frac{\mathbb{Q}\Gamma}{p_{n-i+1}(\mu_i)\mathbb{Q}\Gamma} \hookrightarrow \left(\frac{\mathbb{Q}\Gamma}{p_{n-i+1}(\mu_i)\mathbb{Q}\Gamma} \right) S^{-1}_{p_{n-i+1}}, \tag{7.17}$$

is a monomorphism (using that $p_{n-i+1}(t) \neq 0$ and that μ_i lies in the abelian normal subgroup $A = \pi^{(n-i)}/\pi^{(n-i+1)}_{\mathcal{S}} \subset \Gamma$). This reduces us to showing that

$$\frac{\mathbb{Q}\Gamma}{p_{n-i+1}(\mu_i)\mathbb{Q}\Gamma} \neq 0. \tag{7.18}$$

By the hypotheses of Theorem 5.4, $p_{n-i+1}(t) = q_i(t)$ is not a unit. The map $\mathbb{Z} \to \Gamma$ given by $t \mapsto \mu_i$ is not zero by the inductive hypothesis (7.12). Thus $\langle \mu_i \rangle \subset \Gamma$ and $\mathbb{Q}\Gamma$ is a free $\mathbb{Q}[\mu_i, \mu_i^{-1}]$-module on the cosets of $\langle \mu_i \rangle \in \Gamma$. In the same manner as we showed earlier in the proof, it follows that $p_{n-i+1}(\mu_i)$ is not a unit in the domain $\mathbb{Q}\Gamma$. Therefore (7.18) holds.

This finishes, finally, the inductive step and hence the entire proof of Proposition 7.4, which in turn completes the proofs of (7.2) and (7.3). $\square$

Having established (7.1), (7.2) and (7.3), the proof of Theorem 5.4 is complete.

□

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