APPROXIMATING THE SINE FUNCTION

EXTRA CREDIT FOR MATH 121-01
DUE OCTOBER 18, 2007

The answers to the problems below should be presented neatly, (either typed or written **very** neatly). The graphs should either be printed out or **very** neatly drawn.

You may discuss the problems with other students. However each student is responsible for the final preparation of his or her own paper.

The tangent line to a curve $y = f(x)$ at a point $x = a$ is the **best linear approximation** of the curve near the point $x = a$. That means that the tangent line $y = g_1(x)$ is the line that most resembles the curve near that point, where "most resembles" means that $f(a) = g_1(a)$ and $f'(a) = g_1'(a)$.

Problem 1. *Find the best linear approximation to the curve* $y = \sin x$ *near the point* $x = 0$. *Find the best linear approximation to the curve* $y = \sin x$ *near the point* $x = \pi$.

Problem 2. *Create a graph that includes both* $y = \sin x$ *and the two tangent lines found above.*

Because the tangent line resembles the curve in the way described above, the tangent line can be used to approximate the function $f(x)$ for values of x near $x = a$. In other words, if b is near a, $g_1(b)$ approximates $f(b)$, where $g_1(x)$ is the tangent line to $f(x)$ at $x = a$.

Problem 3. *Use the best linear approximations found above to estimate* $\sin \frac{1}{10}$ *and* $\sin 3$. *(Note that* $\frac{1}{10}$ *is near* 0 *and* 3 *is near* π*.)*

Although the tangent line is the best linear approximation, one can improve the estimate by finding the best quadratic approximation of the curve. The best quadratic approximation of the curve $y = f(x)$ near $x = a$ is a quadratic function $y = g_2(x)$ such that $f(a) = g_2(a)$, $f'(a) = g_2'(a)$, and $f''(a) = g_2''(a)$.

Problem 4. *Find the best quadratic approximation to the curve* $y = \sin x$ *near the point* $x = 0$. *Find the best quadratic approximation to the curve* $y = \sin x$ *near the point* $x = \pi$. *(Hint: To find the best quadratic approximation near* $x = 0$*, start with an arbitrary quadratic function of the form* $g_2(x) = ax^2 + bx + c$*, and then figure out the values of* $a, b,$ *and* c*. To find the best quadratic approximation near* $x = \pi$*, start with an arbitrary quadratic function of the form* $g_2(x) = a(x - \pi)^2 + b(x - \pi) + c$*.)*

Problem 5. *Create a graph that includes both* $y = \sin x$ *and the two quadratic curves found above.*

Since this quadratic function shares more properties with f than the tangent line, it is an even better approximation for f near $x = a$. In other words, if b is near a, $g_2(b)$ approximates $f(b)$ better than $g_1(b)$, where $g_2(x)$ is the best quadratic approximation to $f(x)$ near $x = a$ and $g_1(x)$ is the best linear approximation to $f(x)$ at $x = a$.

Problem 6. *Use the best quadratic approximations found above to estimate* $\sin \frac{1}{10}$ *and* $\sin 3$.

One can get an even better estimate for a curve f, by finding the best cubic approximation to the curve near $x = a$. In other words, the cubic function $y = g_3(x)$ such that $f(a) = g_3(a)$, $f'(a) = g_3'(a)$, $f''(a) = g_3''(a)$, and $f'''(a) = g_3'''(a)$.

Problem 7. *Find the best cubic approximation to the curve* $y = \sin x$ *near the point* $x = 0$. *Find the best cubic approximation to the curve* $y = \sin x$ *near the point* $x = \pi$.

Problem 8. *Create a graph that includes both* $y = \sin x$ *and the two cubic curves found above.*

Problem 9. *Use the best cubic approximations found above to estimate* $\sin\frac{1}{10}$ *and* $\sin 3$.

For any n, one can find the best polynomial of degree n approximation to the curve near $x = a$ by finding the polynomial of degree n, $g_n(x)$, such that $f(a) = g_n(a)$, $f'(a) = g_n'(a)$, $f''(a) = g_n''(a)$, $f'''(a) = g_n'''(a)$,..., $f^{(n)}(a) = g_n^{(n)}(a)$.

Problem 10. *Find the best polynomial of degree* n *approximation to the curve* $y = \sin x$ *near the point* $x = 0$.

Problem 11. *Create one graph that includes* $y = \sin x$ *and the best polynomial of degree* n *approximation to* $y = \sin x$ *near* $x = 0$ *for* $n = 1, 2, 3, 4, 5, 6$.

Problem 12. *Find the best polynomial of degree* n *approximation to the curve* $y = \sin x$ *near the point* $x = \pi$.

Problem 13. *Create one graph that includes* $y = \sin x$ *and the best polynomial of degree* n *approximation to* $y = \sin x$ *near* $x = \pi$ *for* $n = 1, 2, 3, 4, 5, 6$.

A series is an infinite sum of numbers. For example,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

Some series converge, which means that the infinite sum of numbers adds up to a finite number. For example, $\sum\limits_{n=0}^{\infty} \frac{1}{2^n} = 2$.

To be more mathematically precise, an nth partial sum, S_n, of a series is the sum of the first $n + 1$ terms. For the example above,

$$S_0 = 1,\ S_1 = 1 + \frac{1}{2} = \frac{3}{2},\ S_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},\ S_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}, ...$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$

And the definition of the series is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

For the example above,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2.$$

A power series is a function where the value of the function at each point x is a series. For example,

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{x^n} = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + ...$$

In this example, we have already found that $g(2) = 2$. In fact, this function is defined for $x > 1$ because for any such x, the series $\sum\limits_{n=0}^{\infty} \frac{1}{x^n}$ converges. Furthermore, for $x > 1$, it turns out that $\sum\limits_{n=0}^{\infty} \frac{1}{x^n} = \frac{x}{x-1}$. Therefore for $x > 1$ the power series $g(x)$ is the same as the function $\frac{x}{x-1}$. Said another way, a power series expression for $\frac{x}{x-1}$ is $\sum\limits_{n=0}^{\infty} \frac{1}{x^n}$.

The best polynomial of degree n approximations for a function f near $x = a$ can be viewed as expressions for the nth partial sums of a power series expression for f.

Problem 14. *Using the expression for the best polynomial of degree* n *approximation to the curve* $y = \sin x$ *near the point* $x = 0$*, find a power series expression for* $\sin x$.

Problem 15. *Use this power series expression for* $\sin x$ *to find a series that represents* $\sin 1$.