THE FRENET-SERRET FORMULAS

EXTRA CREDIT FOR MATH 222-02
DUE OCTOBER 15, 2009

The answers to the problems below should be presented neatly, (either typed or written **very** neatly).

You may discuss the problems with other students. However each student is responsible for the final preparation of his or her own paper.

Before completing this project, it may be helpful to carefully read Section 14.3.

Suppose $\mathbf{r}(t)$ describes a smooth space curve. The *unit tangent vector* to the curve $\mathbf{r}(t)$ is defined to be:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The *principal unit normal vector* is defined to be:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

The *binormal vector* is defined to be:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

The triple $(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))$ is referred to as the *Frenet frame.*

Problem 1. *Give a short proof to show that for each t, $\mathbf{T}(t)$ is orthogonal to $\mathbf{N}(t)$. Why is $\mathbf{B}(t)$ orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$?*

Problem 2. *Find the Frenet frame for the* hyperbolic helix*:*

$$\mathbf{r}(t) = \langle \cosh t, \sinh t, t \rangle.$$

Here

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2}$$

are the hyperbolic cosine and hyperbolic sine functions.

The arclength function $s(t)$ is defined to be:

$$s(t) = \int_0^t |\mathbf{r}'(u)| \, du.$$

Using the arclength function, one can write t as a function of s, in order to reparametrize the curve by arclength: $\mathbf{r}(s) \equiv \mathbf{r}(t(s))$.

Problem 3. *Find the arclength function for the hyperbolic helix. Use it to reparametrize the curve by arclength.*

By the definition of the arclength function $s(t)$ and the Fundamental Theorem of Calculus, we have that

$$\frac{ds}{dt} = s'(t) = |\mathbf{r}'(t)|.$$

Therefore by the Chain Rule,

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'(s)\,|\mathbf{r}'(t)|.$$

In other words,

$$\mathbf{r}'(s) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Therefore, $|\mathbf{r}'(s)| = 1$.

Problem 4. *Show that if $\mathbf{r}(s)$ is parametrized by arclength, then $\mathbf{T}(s) = \mathbf{r}'(s)$.*

From now on, let's assume that $\mathbf{r}(s)$ is always parametrized by arclength. The *curvature* of $\mathbf{r}$ is defined to be:

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|.$$

Problem 5. *Find the curvature of the hyperbolic helix.*

Problem 6. *Show that for any curve* $\mathbf{r}(s)$ *parametrized by arclength,*

$$\mathbf{T}'(s) = \kappa \mathbf{N}(\mathbf{s}).$$

Since $\mathbf{T}$ is orthogonal to $\mathbf{B}$, we have $\mathbf{T} \cdot \mathbf{B} = 0$. Differentiating both sides of this we have:

$$\mathbf{T}' \cdot \mathbf{B} + \mathbf{T} \cdot \mathbf{B}' = 0.$$

Problem 7. *Continue this argument to show that* $\mathbf{B}'$ *is orthogonal to* $\mathbf{T}$.

Problem 8. *Find a similar argument to show that* $\mathbf{B}'$ *is orthogonal to* $\mathbf{B}$.

Since $\mathbf{B}'$ is orthogonal to both $\mathbf{T}$ and $\mathbf{B}$, it must be parallel to $\mathbf{N}$. Define the *torsion* of $\mathbf{r}$ to be:

$$\tau = -\mathbf{N} \cdot \mathbf{B}'.$$

Problem 9. *Find the torsion of the hyperbolic helix.*

Problem 10. *Show that for any curve* $\mathbf{r}(s)$ *parametrized by arclength,*

$$\mathbf{B}' = -\tau \mathbf{N}.$$

Problem 11. *Show that* $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

Problem 12. *Use the above formula to show that* $\mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B}$.

The formulas in Problems 6, 10, and 12 are called the *Frenet-Serret formulas.*