TRIPLE INTEGRALS

EXTRA CREDIT FOR MATH 222
DUE APRIL 26, 2013

There are two parts to this extra credit assignment. Together these two parts can increase your final course grade by 2 percentage points.

Part 1 is given below. You should turn in the answers to Part 1 by April 26 to my office (Exley 641) - by sliding it under the door, if necessary.

Part 2 is a brief timed quiz on triple integrals that will be given, Friday, April 26, 2:00-2:30pm. If you cannot take the quiz at that time, please email me by Wednesday, April 24 to make other arrangements.

Part 1

The answers to the problems below should be presented neatly - either typed or written **very** neatly. You may discuss the problems with other students. However each student is responsible for the final preparation of his or her own paper. Before completing this project, you should carefully read sections 15.7 through 15.9 in the textbook.

Suppose $f(x, y)$ is a function of two variables, and $R = [a, b] \times [c, d]$ is a rectangle in the plane. To define the double integral of f over the rectangle R, we first subdivided the rectangle R into smaller subrectangles R_{ij} by subdividing the interval $[a, b]$ into m subintervals, each of length Δx and subdividing the interval $[c, d]$ into n subintervals, each of length Δy. Therefore each subrectangle R_{ij} had dimensions $\Delta x \times \Delta y$. In particular, the area of each R_{ij} was $\Delta A \equiv \Delta x \Delta y$. Then for each subrectangle, we chose a point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$. An estimate for the double integral of f over R was given by $\Sigma_{i=1}^m \Sigma_{j=1}^n f(x^*_{ij}, y^*_{ij})\Delta A$. Finally, we subdivided R into smaller and smaller subrectangles and defined the double integral to be the following limit:

$$\iint_R f(x, y)\, dA \equiv \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\, \Delta A,$$

if the limit exists.

Now suppose $f(x, y, z)$ is a function of *three* variables, and $B = [a, b] \times [c, d] \times [r, s]$ is a *box* in 3-space. We want to define the *triple integral* of $f(x, y, z)$ over B, in a similar manner. We begin by subdividing the box B into sub-boxes B_{ijk} by subdividing $[a, b]$ into l subintervals (each of length Δx), $[c, d]$ into m subintervals (each of length Δy), and $[r, s]$ into n subintervals (each of length Δz). Therefore each B_{ijk} has dimensions $\Delta x \times \Delta y \times \Delta z$ and has *volume* $\Delta V \equiv \Delta x \Delta y \Delta z$. Next, for each sub-box, we choose a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in B_{ijk}$. We define the *triple integral* of $f(x, y, z)$ over the box B to be the following limit:

$$\iiint_B f(x, y, z)\, dV \equiv \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V,$$

if the limit exists.

The goal of this extra credit project is to learn to evaluate triple integrals by using techniques that are generalizations of those we used to evaluate double integrals.

One important tool that we used to evaluate double integrals was Fubini's Theorem. Luckily, this generalizes to triple integrals.

Theorem. *If f is continuous and $B = [a, b] \times [c, d] \times [r, s]$ is a box, then*

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

As was the case for double integrals, Fubini's Theorem works for any order of integration in the iterated integral.

Problem 1. *Evaluate the integral* $\iiint_B (xz - y^3)\, dV$, *where* $B = [-1, 1] \times [0, 2] \times [0, 1]$.

Suppose $f(x, y, z)$ is a function of three variables and E is a bounded region in 3-space (but not necessarily a box). To define the triple integral of f over E, we begin by choosing a box B that contains the region E. Then we define a new function $F(x, y, z)$ on B as follows:

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \notin E. \end{cases}$$

Then we define $\iiint_E f(x, y, z)\, dV \equiv \iiint_B F(x, y, z)\, dV$. (This process is similar to how we defined double integrals over general regions in the plane.)

Suppose E is a region of the following special form:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\},$$

for some region D in the plane and some continuous functions u_1 and u_2 of two variables. If we choose a rectangle $R = [a, b] \times [c, d]$ that contains D and numbers r and s so that $r \le u_1(x, y) \le u_2(x, y) \le s$ for all $(x, y) \in D$, then the box $B = R\times[r, s] = [a, b]\times[c, d]\times[r, s]$ contains E. Therefore $\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$, where F is defined as above.

By Fubini's Theorem, this is the same as the iterated integral $\int_c^d \int_a^b \int_r^s F(x, y, z)\, dz\, dx\, dy$. Notice that when we evaluate the innermost integral, we treat x and y as constants. Once we have evaluated this innermost integral, we are then integrating a function of just x and y. If $(x, y) \notin D$, then $(x, y, z) \notin E$, and therefore $F(x, y, z) = 0$ and hence $\int_r^s F(x, y, z)\, dz = 0$. Therefore,

$$\int_c^d \int_a^b \int_r^s F(x,y,z)\, dz\, dx\, dy = \iint_D \int_r^s F(x,y,z)\, dz\, dA.$$

Furthermore, since $F(x, y, z)$ is either $f(x, y, z)$ or 0, depending on whether $(x, y, z) \in E$ or not, for a fixed (x, y), we have:

$$F(x,y,z) = \begin{cases} 0 & \text{if } z < u_1(x,y) \\ f(x,y,z) & \text{if } u_1(x,y) \le z \le u_2(x,y) \\ 0 & \text{if } z > u_2(x,y). \end{cases}$$

Thus,

$$\iint_D \int_r^s F(x,y,z)\, dz\, dA = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dA.$$

Putting this all together, we have that if the region E is of the special form above, then

$$\iiint_E f(x,y,z)\, dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dA.$$

Of course, similar formulas result from interchanging the variables.

Problem 2. *Use the formula above, to evaluate* $\iiint_E yz \cos x^5\, dV$, *where* $E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ x \le z \le 2x\}$.

Problem 3. *Evaluate* $\iiint_E xy\, dV$, *where E is the region bounded by the parabolic cylinders* $y = x^2$ *and* $x = y^2$ *and the planes* $z = 0$ *and* $z = x + y$.

Problem 4. *Evaluate* $\iiint_E z\, dV$, *where E is the region bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0$, $y = 3x$, *and* $z = 0$ *in the first octant.*

Recall that the area of a region D in the plane can be computed as $\iint_D dA$. Similarly, the volume of a region E in 3-space can be computed as $\iiint_E dV$.

Problem 5. *Find the volume of the solid bounded by the cylinder* $y = x^2$ *and the planes* $z = 0$, $z = 4$, *and* $y = 9$.

When evaluating double integrals, we found that it was sometimes easier to describe a region in the plane using polar coordinates (r, θ) rather than the usual Cartesian coordinates, (x, y). Similarly, when describing regions in 3-space, it is sometimes easier to use a different coordinate system.

A simple 3-dimensional coordinate system results from using polar coordinates (r, θ) to describe the location of the projection of a point in 3-space onto the xy-plane followed by the usual z-coordinate to describe the height of the point. Therefore a point in 3-space is specified by (r, θ, z). This coordinate system is known as *cylindrical coordinates*.

When evaluating double integrals, we found that if $D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$, then $\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$. (Note the extra r.)

Combining this formula with the one on the previous page, we have that if

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

where

$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\},$$

then

$$\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Problem 6. *Use cylindrical coordinates to evaluate* $\iiint_E x\ dV$, *where* E *is the region enclosed by the planes* $z = 0$ *and* $z = x + y + 5$ *and by the cylinders* $x^2 + y^2 = 4$ *and* $x^2 + y^2 = 9$.

Problem 7. *Find the volume of the solid that lies within both the cylinder* $x^2 + y^2 = 1$ *and the sphere* $x^2 + y^2 + z^2 = 4$.

Problem 8. *Evaluate the integral by changing to cylindrical coordinates:*

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx.$$

Section 15.9 of the textbook describes another 3-dimensional coordinate system called *spherical coordinates.*

Problem 9. *Write a brief explanation of spherical coordinates. Specifically, include an explanation of where the point with spherical coordinates* $(4, 3\pi/4, \pi/3)$ *is located. Also include an explanation of what the spherical coordinates are of the point with Cartesian coordinates* $(0, \sqrt{3}, 1)$.

Problem 10. *Briefly explain why the formula on page 1059 holds.*

Problem 11. *Use the formula to evaluate* $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\ dV$, *where* E *is the region enclosed by the sphere* $x^2 + y^2 + z^2 = 9$ *in the first octant.*

Problem 12. *Find the volume of the solid that lies within the sphere* $x^2 + y^2 + z^2 = 4$, *above the xy-plane, and below the cone* $z = \sqrt{x^2+y^2}$.