

## Problem Session Problems

Math 244

April 12, 2010

1. Recall that the *comb space* is the subspace of  $\mathbb{R}^2$  that is the union of  $[0, 1] \times \{0\}$ ,  $\{0\} \times [0, 1]$ , and  $\{\frac{1}{n}\} \times [0, 1]$  for all  $n = 1, 2, 3, \dots$ 
  - (a) Show that the comb space is path-connected.
  - (b) If we remove the point  $(\frac{2}{3}, 0)$  from the comb space is the resulting space still path-connected? (Recall that last week we showed that the resulting space is *not* connected.)
  - (c) If we remove the point  $(0, 0)$  from the comb space is the resulting space still path-connected? (Recall that last week we showed that the resulting space *is* connected.)

2. Recall that a space  $X$  is *locally connected* if for each  $x \in X$ , and each neighborhood  $U$  of  $x$ , there is a connected neighborhood  $V$  of  $x$  which is contained in  $U$ .

Similarly, a space  $X$  is *locally path-connected* if for each  $x \in X$ , and each neighborhood  $U$  of  $x$ , there is a path-connected neighborhood  $V$  of  $x$  which is contained in  $U$ .

- (a) Show that if a space is locally path-connected, then it is locally connected.
- (b) Let  $A = \{(x, y) \mid y = \sin \frac{1}{x}, 0 < x \leq 1\}$ . Let  $X$  be the union of  $A$  with the four line segments  $\{1\} \times [-2, \sin 1]$ ,  $[-1, 1] \times \{-2\}$ ,  $\{-1\} \times [-2, 0]$ , and  $[-1, 0] \times \{0\}$ . Show that  $X$  is not locally path-connected, but is locally connected.
- (c) Show that the comb space is path-connected, but is not locally path-connected.
- (d) Show that if a space is connected and locally path-connected, then it is path-connected.

Note: The last question is on the next page.

3. For each of the combinations below, either find an example that has those properties or explain why one does not exist.

Example	Connected	Path-connected	Locally Connected	Locally Path-Connected
	Yes	Yes	Yes	Yes
	Yes	Yes	Yes	No
	Yes	Yes	No	Yes
	Yes	Yes	No	No
	Yes	No	Yes	Yes
	Yes	No	Yes	No
	Yes	No	No	Yes
	Yes	No	No	No
	No	Yes	Yes	Yes
	No	Yes	Yes	No
	No	Yes	No	Yes
	No	Yes	No	No
	No	No	Yes	Yes
	No	No	Yes	No
	No	No	No	Yes
	No	No	No	No