Problem Session Problems

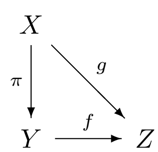
Math 244

April 26, 2010

1. Recall the following theorem from class.

   Theorem. *Let X and Z be topological spaces and let $g : X \to Z$ be a continuous and surjective map. Define an equivalence relation on X as follows: $x \sim y$ if and only if $g(x) = g(y)$. Let Y be the identification space resulting from this equivalence relation and let $\pi : X \to Y$ be the identification map (i.e. $\pi(x) = [x]$).*

   Then there is a bijective and continuous map $f : Y \to Z$ such that $g = f \circ \pi$.

   Furthermore, f is a homeomorphism if and only if g is an identification map (i.e. U is open in Z if and only if $g^{-1}(U)$ is open in X.

   We will use this theorem to show that S^1 is homeomorphic to the identification space of $I = [0, 1]$ where $0 \sim 1$.

   (a) Recall that $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. Define a surjective, continuous map $g : I \to S^1$ such that if $g(a) = g(b)$, then either $a = b$ or $a, b \in \{0, 1\}$.

   (b) Show that g is an identification map.

   (c) Use the above theorem to show that S^1 is homeomorphic to the identification space of I where $0 \sim 1$.

2. The *Hawaiian earring* is the topological space

   $$X = \bigcup_{n=1}^{\infty} \left\{ (x, y) \middle| \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\} \subset \mathbb{R}^2$$

   endowed with the subspace topology. Let Y be the identification space of the real line that results from identifying all the integers to a single point.

   Show that X and Y are not homeomorphic.

   (Hint: If $h : X \to Y$ is a homeomorphism, then $h|_{X - x_0} : X - x_0 \to Y - h(x_0)$ is a homeomorphism.)